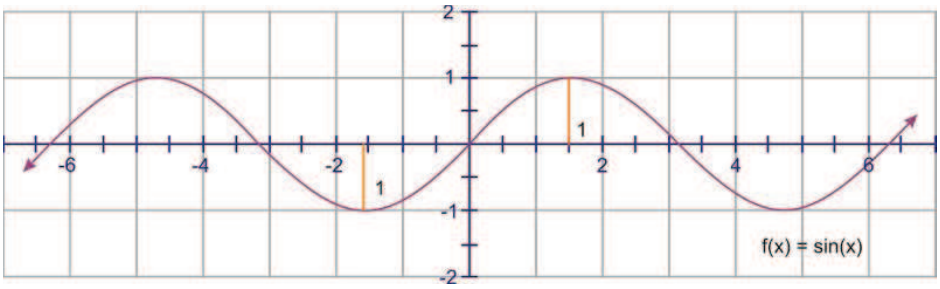
2.5 Amplitude, Period and Frequency

Learning Objectives

- Calculate the amplitude and period of a sine or cosine curve.
- Calculate the frequency of a sine or cosine wave.
- Graph transformations of sine and cosine waves involving changes in amplitude and period (frequency).

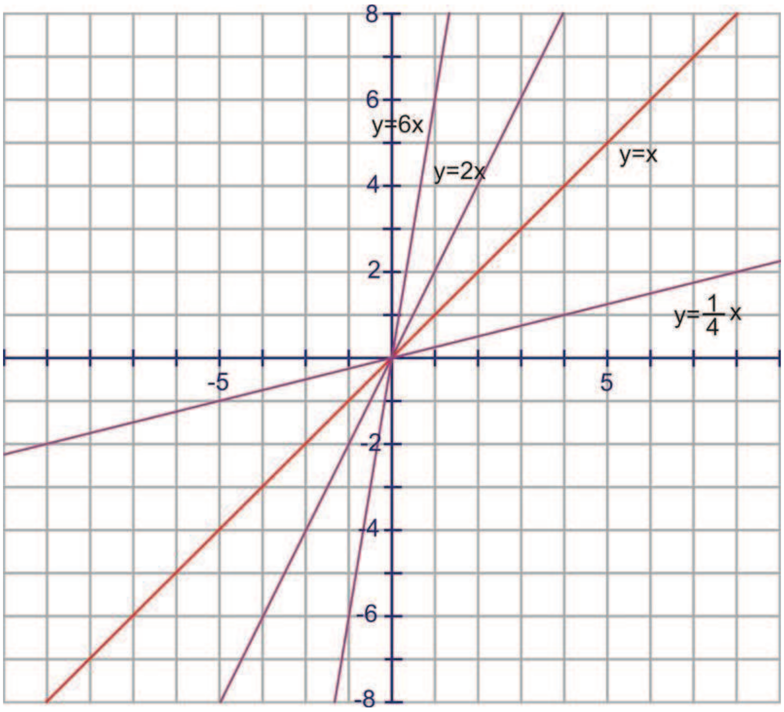
Amplitude

The **amplitude** of a wave is basically a measure of its height. Because that height is constantly changing, amplitude is defined as the *farthest* distance the wave gets from its center. In a graph of $f(x) = \sin x$, the wave is centered on the $x-$axis and the farthest away it gets (in either direction) from the axis is 1 unit.



So the amplitude of $f(x) = \sin x$ (and $f(x) = \cos x$) is 1.

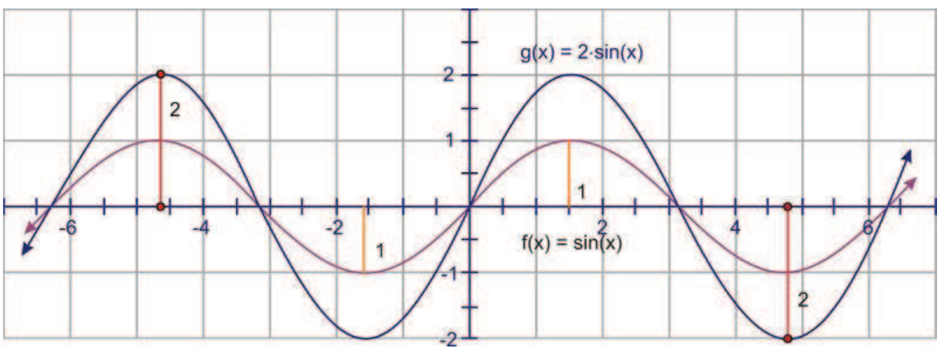
Recall how to transform a linear function, like $y = x$. By placing a constant in front of the x value, you may remember that the slope of the graph affects the steepness of the line.



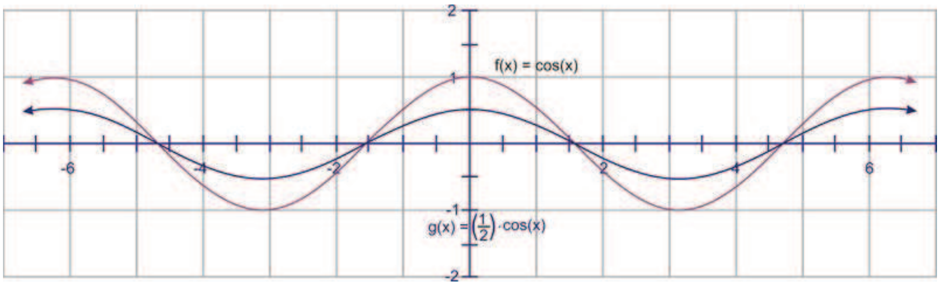
The same is true of a parabolic function, such as $y = x^2$. By placing a constant in front of the x^2, the graph would be either wider or narrower. So, a function such as $y = \frac{1}{8}x^2$, has the same parabolic shape but it has been "smooshed," or looks wider, so that it increases or decreases at a lower rate than the graph of $y = x^2$.

No matter the basic function; linear, parabolic, or trigonometric, the same principle holds. To dilate (flatten or steepen, wide or narrow) the function, multiply the function by a constant. Constants greater than 1 will stretch the graph vertically and those less than 1 will shrink it vertically.

Look at the graphs of $y = \sin x$ and $y = 2\sin x$.

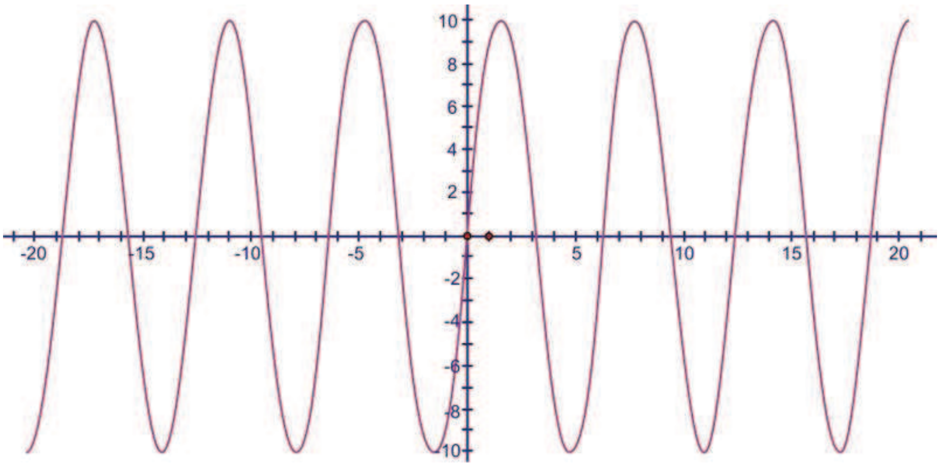


Notice that the amplitude of $y = 2\sin x$ is now 2. An investigation of some of the points will show that each $y-$value is twice as large as those for $y = \sin x$. Multiplying values less than 1 will decrease the amplitude of the wave as in this case of the graph of $y = \frac{1}{2}\cos x$:



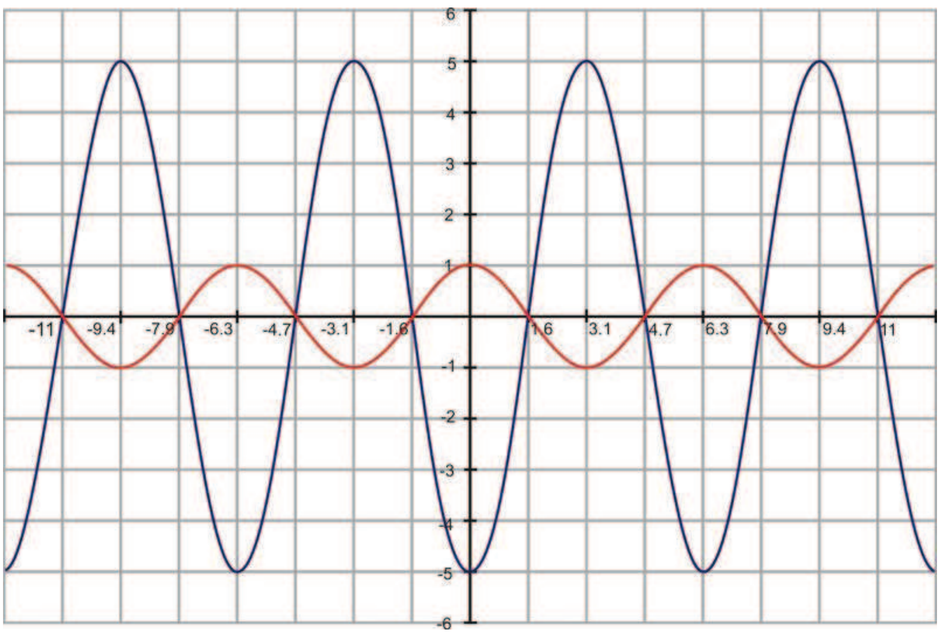
Example 1: Determine the amplitude of $f(x) = 10\sin x$.

Solution: The 10 indicates that the amplitude, or height, is 10. Therefore, the function rises and falls between 10 and -10.



Example 2: Graph $g(x) = -5\cos x$

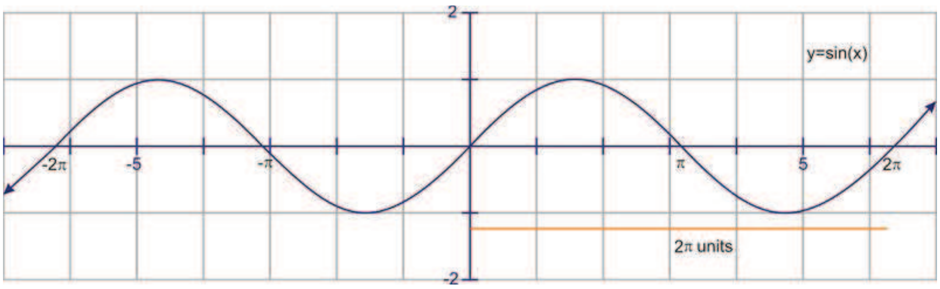
Solution: Even though the 5 is negative, the amplitude is still positive 5. The amplitude is always the absolute value of the constant A. However, the negative changes the appearance of the graph. Just like a parabola, the sine (or cosine) is flipped upside-down. Compare the blue graph, $g(x) = -5\cos x$, to the red parent graph, $f(x) = \cos x$.



So, in general, the constant that creates this stretching or shrinking is the amplitude of the sinusoid. Continuing with our equations from the previous section, we now have $y = D \pm A\sin(x \pm C)$ or $y = D \pm A\cos(x \pm C)$. Remember, if $0 < |A| < 1$, then the graph is shrunk and if $|A| > 1$, then the graph is stretched. And, if A is negative, then the graph is flipped.

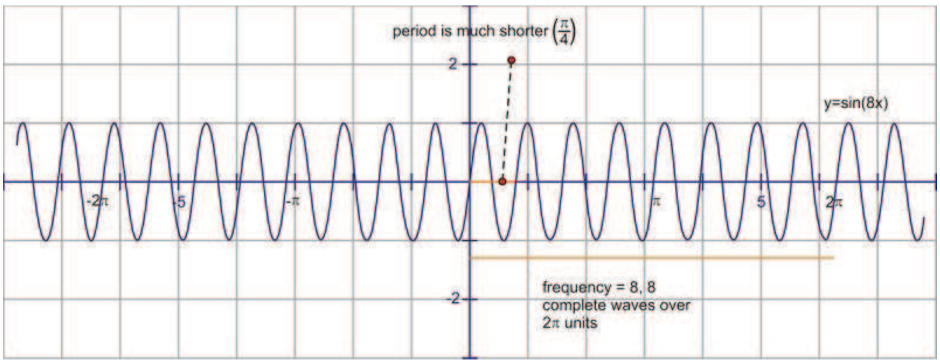
Period and Frequency

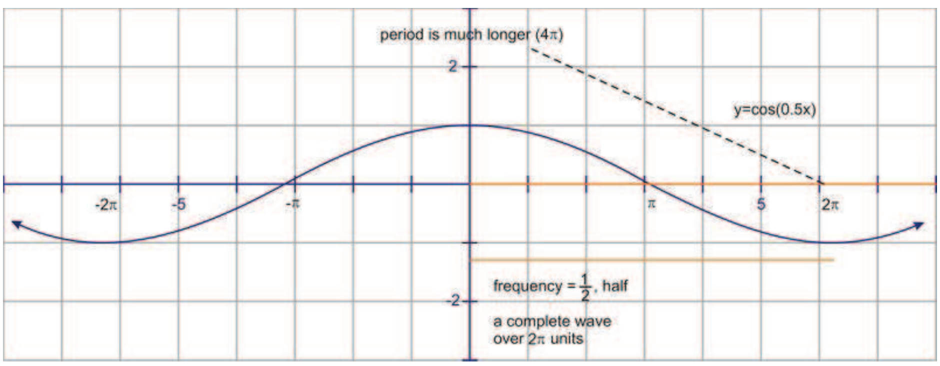
The **period** of a trigonometric function is the horizontal distance traversed before the $y-$values begin to repeat. For both graphs, $y = \sin x$ and $y = \cos x$, the period is 2π. As we learned earlier in the chapter, after completing one rotation of the unit circle, these values are the same.



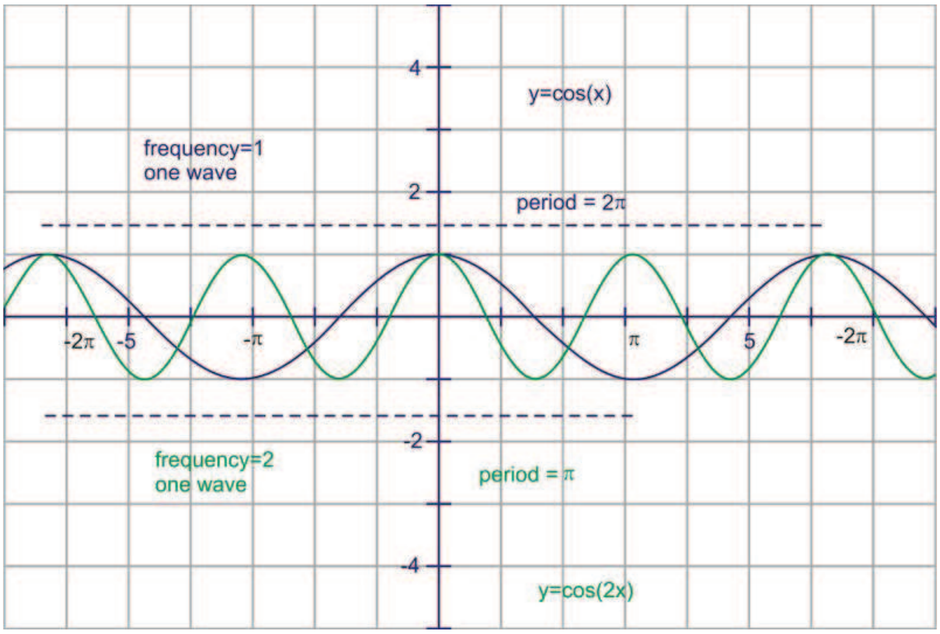
Frequency is a measurement that is closely related to period. In science, the frequency of a sound or light wave is the number of complete waves for a given time period (like seconds). In trigonometry, because all of these periodic functions are based on the unit circle, we usually measure frequency as the number of complete waves every 2π units. Because $y = \sin x$ and $y = \cos x$ cover exactly one complete wave over this interval, their frequency is 1.

Period and frequency are inversely related. That is, the higher the frequency (more waves over 2π units), the lower the period (shorter distance on the $x-$axis for each complete cycle).





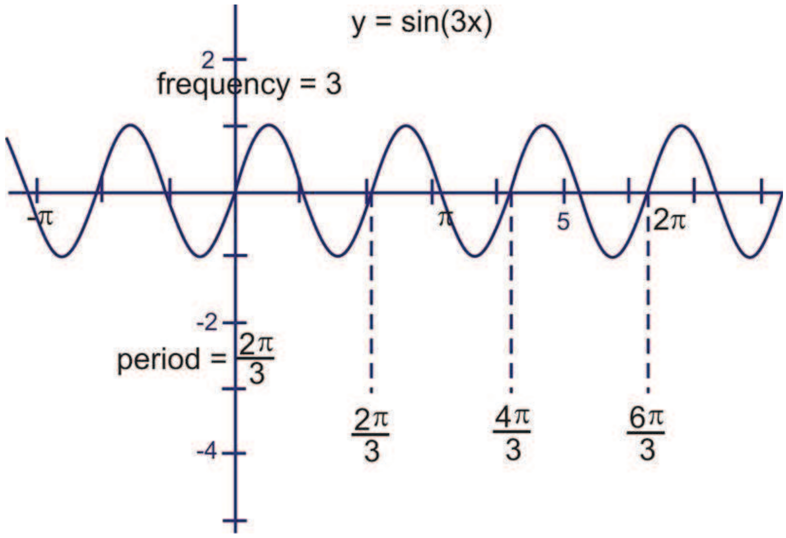
After observing the transformations that result from multiplying a number *in front of* the sinusoid, it seems natural to look at what happens if we multiply a constant *inside* the argument of the function, or in other words, by the x value. In general, the equation would be $y = \sin Bx$ or $y = \cos Bx$. For example, look at the graphs of $y = \cos 2x$ and $y = \cos x$.



Notice that the number of waves for $y = \cos 2x$ has increased, in the same interval as $y = \cos x$. There are now **2 waves** over the interval from 0 to 2π. Consider that you are doubling each of the x values because the function is $2x$. When π is plugged in, for example, the function becomes 2π. So the portion of the graph that normally corresponds to 2π units on the $x-$axis, now corresponds to *half* that distance—so the graph has been "scrunched" horizontally. The frequency of this graph is therefore 2, or the same as the constant we multiplied by in the argument. The period (the length for each complete wave) is π.

Example 3: What is the frequency and period of $y = \sin 3x$?

Solution: If we follow the pattern from the previous example, multiplying the angle by 3 should result in the sine wave completing a cycle **three times** as often as $y = \sin x$. So, there will be three complete waves if we graph it from 0 to 2π. The frequency is therefore 3. Similarly, if there are 3 complete waves in 2π units, one wave will be a third of that distance, or $\frac{2\pi}{3}$ radians. Here is the graph:



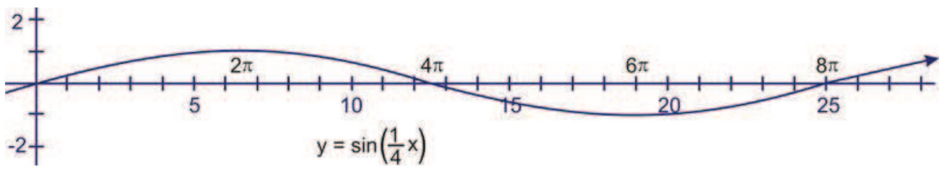
This number that is multiplied by x, called B, will create a horizontal dilation. The larger the value of B, the more compressed the waves will be horizontally. To stretch out the graph horizontally, we would need to *decrease* the frequency, or multiply by a number that is less than 1. Remember that this dilation factor is *inversely* related to the period of the graph.

Adding, one last time to our equations from before, we now have: $y = D \pm A\sin(B(x \pm C))$ or $y = D \pm A\cos(B(x \pm C))$, where B is the frequency, the period is equal to $\frac{2\pi}{B}$, and everything else is as defined before.

Example 4: What is the frequency and period of $y = \cos\frac{1}{4}x$?

Solution: Using the generalization above, the frequency must be $\frac{1}{4}$ and therefore the period is $\frac{2\pi}{\frac{1}{4}}$, which simplifies to: $\frac{2\pi}{\frac{1}{4}} = \frac{\frac{2\pi}{1}}{\frac{1}{4}} \cdot \frac{\frac{4}{1}}{\frac{4}{1}} = \frac{8\pi}{1} = 8\pi$

Thinking of it as a transformation, the graph is stretched horizontally. We would only see $\frac{1}{4}$ of the curve if we graphed the function from 0 to 2π. To see a complete wave, therefore, we would have to go four times as far, or all the way from 0 to 8π.

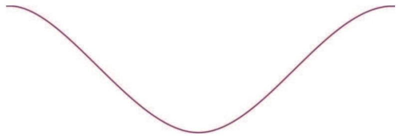


Combining Amplitude and Period

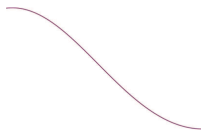
Here are a few examples with both amplitude and period.

Example 5: Find the period, amplitude and frequency of $y = 2\cos\frac{1}{2}x$ and sketch a graph from 0 to 2π.

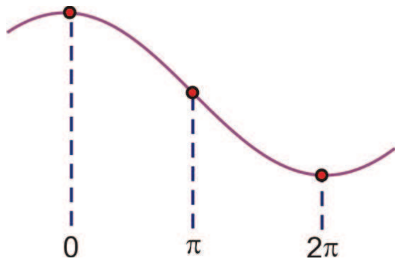
Solution: This is a cosine graph that has been stretched both vertically and horizontally. It will now reach up to 2 and down to -2. The frequency is $\frac{1}{2}$ and to see a complete period we would need to graph the interval $[0, 4\pi]$. Since we are only going out to 2π, we will only see half of a wave. A complete cosine wave looks like this:



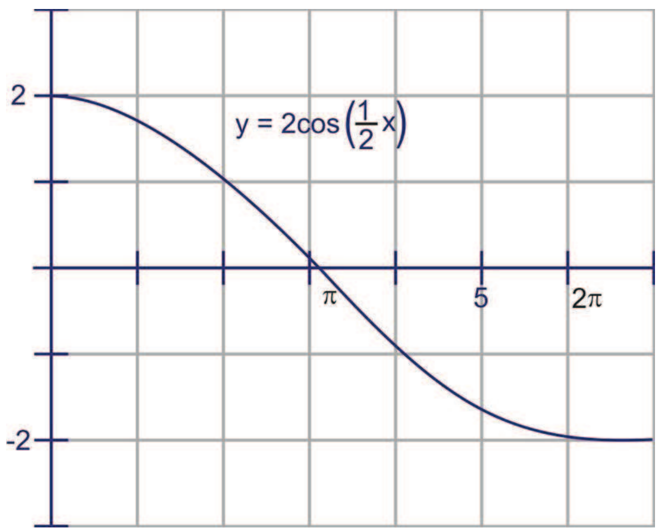
So, half of it is this:



This means that this half needs to be stretched out so it finishes at 2π, which means that at π the graph should cross the $x-$axis:

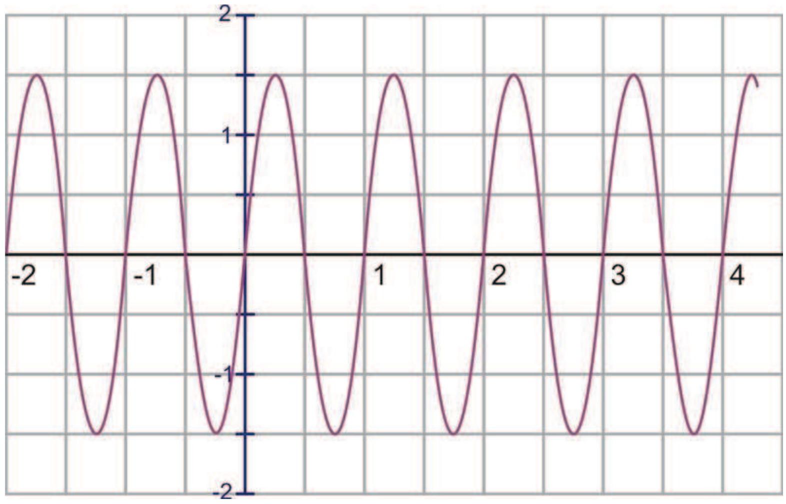


The final sketch would look like this:



amplitude $= 2,$ frequency $= \frac{1}{2},$ period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$

Example 6: Identify the period, amplitude, frequency, and equation of the following sinusoid:



Solution: The amplitude is 1.5. Notice that the units on the $x-$axis are not labeled in terms of π. This appears to be a sine wave because the $y-$intercept is 0.

One wave appears to complete in 1 unit (*not* 1π *units!*), so the period is 1. If one wave is completed in 1 unit, how many waves will be in 2π units? In previous examples, you were given the frequency and asked to find the period using the following relationship:

$$p = \frac{2\pi}{B}$$

Where B is the frequency and p is the period. With just a little bit of algebra, we can transform this formula and solve it for B:

$$p = \frac{2\pi}{B} \rightarrow Bp = 2\pi \rightarrow B = \frac{2\pi}{p}$$

Therefore, the frequency is:

$$B = \frac{2\pi}{1} = 2\pi$$

If we were to graph this out to 2π we would see 2π (or a little more than 6) complete waves.

Replacing these values in the equation gives: $f(x) = 1.5\sin 2\pi x$.

Points to Consider

- Are the graphs of the other four trigonometric functions affected in the same way as sine and cosine by amplitude and period?
- We saw what happens to a graph when A is negative. What happens when B is negative?

Review Questions

1. Using the graphs from section 3, identify the period and frequency of $y = \sec x, y = \cot x$ and $y = \csc x$.
2. Identify the minimum and maximum values of these functions.

   a. $y = \cos x$
   b. $y = 2\sin x$
   c. $y = -\sin x$
   d. $y = \tan x$
   e. $y = \frac{1}{2}\cos 2x$
   f. $y = -3\sin 4x$

3. How many real solutions are there for the equation $4\sin x = \sin x$ over the interval $0 \le x \le 2\pi$?

   a. 0
   b. 1
   c. 2
   d. 3

4. For each equation, identify the period, amplitude, and frequency.

   a. $y = \cos 2x$
   b. $y = 3\sin x$
   c. $y = 2\sin \pi x$
   d. $y = 2\cos 3x$
   e. $y = \frac{1}{2}\cos \frac{1}{2}x$
   f. $y = 3\sin \frac{1}{2}\,x$

5. For each of the following graphs; 1) identify the period, amplitude, and frequency and 2) write the equation.

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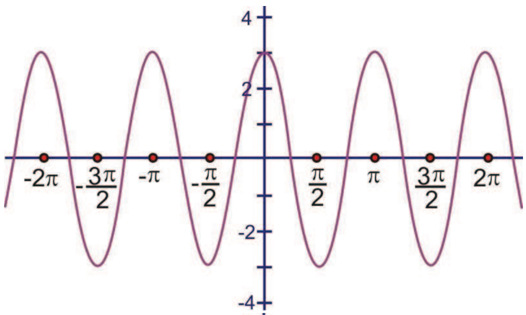

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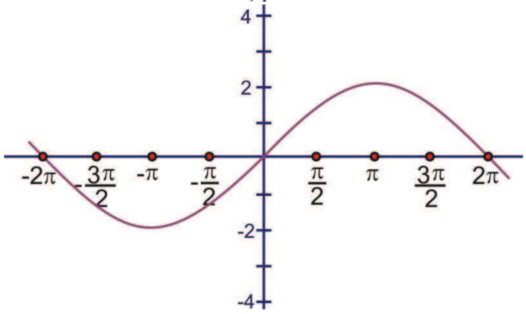
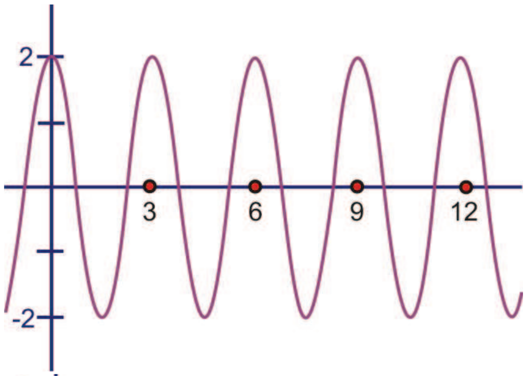
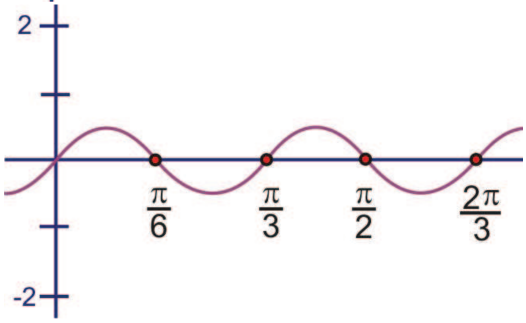




c.



d.

6. For each equation, draw a sketch from 0 to 2π.

   a. $y = 3\sin 2x$
   b. $y = 2.5\cos \pi x$
   c. $y = 4\sin \frac{1}{2}x$

Review Answers

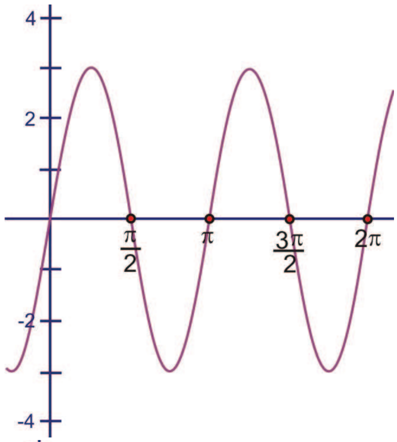
1. $y = \sec x$: period $= 2\pi$, frequency $= 1$ $y = \cot x$: period $= \pi$, frequency $= 2$ $y = \csc x$: period $= 2\pi$, frequency $= 1$ Because these are reciprocal functions, the periods are the same as cosine, tangent, and sine, repectively.

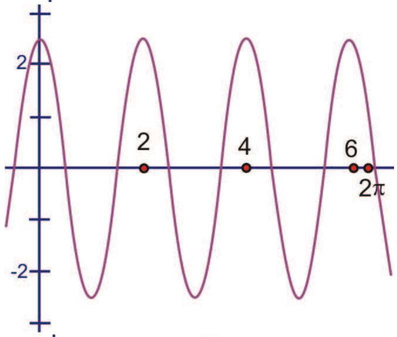
   1. min: -1, max: 1
   2. min: -2, max: 2
   3. min: -1, max: 1
   4. there is no minimum or maximum, tangent has a range of all real numbers
   5. min: $-\frac{1}{2}$, max: $\frac{1}{2}$
   6. min: -3, max: 3

2. d.

   1. period: π, amplitude: 1, frequency: 2
   2. period: 2π, amplitude: 3, frequency: 1
   3. period: 2, amplitude: 2, frequency: π
   4. period: $\frac{2\pi}{3}$, amplitude: 2, frequency: 3
   5. period: 4π, amplitude: $\frac{1}{2}$, frequency: $\frac{1}{2}$
   6. period: 4π, amplitude: 3, frequency: $\frac{1}{2}$

   1. period: π, amplitude: 1, frequency: $2, y = 3\cos 2x$
   2. period: 4π, amplitude: 2, frequency: $\frac{1}{2}, y = 2\sin \frac{1}{2}x$
   3. period: 3, amplitude: 2, frequency: $\frac{2\pi}{3}, y = 2\cos \frac{2\pi}{3}x$
   4. period: $\frac{\pi}{3}$, amplitude: $\frac{1}{2}$, frequency: $6, y = \frac{1}{2}\sin 6x$

1. 

2. 

3.