2.2 Derivatives: Properties and Formulas

This section begins with a look at which functions have derivatives. Then we'll examine how to calculate derivatives of elementary combinations of basic functions. By knowing the derivatives of some basic functions and just a few differentiation patterns, you will be able to calculate the derivatives of a tremendous variety of functions. This section contains most—but not quite all—of the general differentiation patterns you will ever need.

Which Functions Have Derivatives?

A function must be continuous in order to be differentiable.

It is vital to understand what this theorem tells us and what it does **not** tell us: If a function is differentiable at a point, then the function is automatically continuous there. If the function is continuous at a point, then the function may or may not be differentiable there.

> **Theorem**:
>
> If a function is differentiable at a point
> then it is continuous at that point.

Proof. Assume that the hypothesis (f is differentiable at the point c) is true. Then $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$ must exist and be equal to $f'(c)$. We want to show that f must necessarily be continuous at c, so we need to show that $\lim_{h\to 0} f(c+h) = f(c)$.

It's not yet obvious why we want to do so, but we can write:

$$f(c+h) = f(c) + \frac{f(c+h)-f(c)}{h} \cdot h$$

and then compute the limit of both sides of this expression:

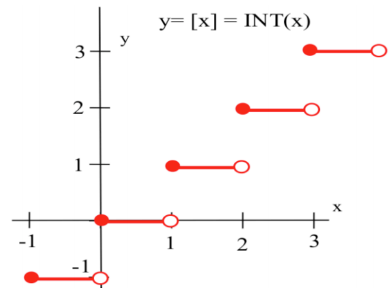
$$\begin{aligned}
\lim_{h\to 0} f(c+h) &= \lim_{h\to 0} f(c) + \frac{f(c+h)-f(c)}{h} \cdot h \\
&= \lim_{h\to 0} f(c) + \lim_{h\to 0} \left(\frac{f(c+h)-f(c)}{h} \cdot h \right) \\
&= \lim_{h\to 0} f(c) + \lim_{h\to 0} \left(\frac{f(c+h)-f(c)}{h} \right) \cdot \lim_{h\to 0} h \\
&= f(c) + f'(c) \cdot 0 = f(c)
\end{aligned}$$

Therefore f is continuous at c. □

We often use the contrapositive form of this theorem, which tells us about some functions that do **not** have derivatives.

> **Contrapositive Form of the Theorem**:
>
> If f is not continuous at a point
> then f is not differentiable at that point.

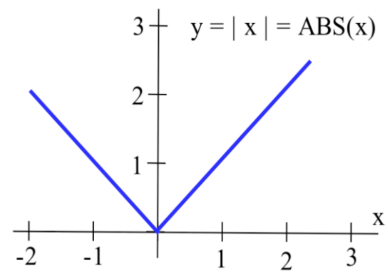
Example 1. Show that $f(x) = \lfloor x \rfloor$ is not continuous and not differentiable at 2 (see margin figure).



Solution. The one-sided limits $\lim_{x\to 2^+} \lfloor x \rfloor = 2$ and $\lim_{x\to 2^-} \lfloor x \rfloor = 1$ have different values, so $\lim_{x\to 2} \lfloor x \rfloor$ does not exist. Therefore $f(x) = \lfloor x \rfloor$ is not continuous at 2, and as a result it is not differentiable at 2. ◀

Lack of continuity implies lack of differentiability, but the next examples show that continuity is not enough to guarantee differentiability.

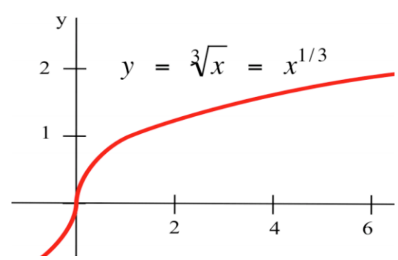
Example 2. Show that $f(x) = |x|$ is continuous but not differentiable at $x = 0$ (see margin figure).



Solution. We know that $\lim_{x\to 0} |x| = 0 = |0|$, so f is continuous at 0, but in Section 2.1 we saw that $|x|$ was not differentiable at $x = 0$. ◀

> A function is not differentiable at a cusp or a "corner."

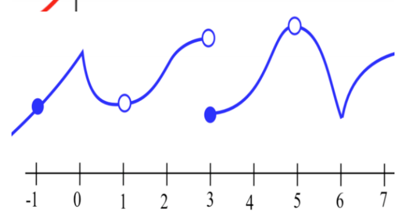
Example 3. Show that $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ is continuous but not differentiable at $x = 0$ (see margin figure).



Solution. We can verify that $\lim_{x\to 0^+} \sqrt[3]{x} = \lim_{x\to 0^-} \sqrt[3]{x} = 0$, so $\lim_{x\to 0} \sqrt[3]{x} = 0 = \sqrt[3]{0}$ so f is continuous at 0. But $f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$, which is undefined at $x = 0$, so f is not differentiable at 0. ◀

> A function is not differentiable where its tangent line is vertical.

Practice 1. At which integer values of x is the graph of f in the margin figure continuous? Differentiable?



Graphically, a function is **continuous** if and only if its graph is "connected" and does not have any holes or breaks. Graphically, a function is **differentiable** if and only if it is continuous and its graph is "smooth" with no corners or vertical tangent lines.

Derivatives of Elementary Combinations of Functions

We now begin to compute derivatives of more complicated functions built from combinations of simpler functions.

Example 4. The derivative of $f(x) = x$ is $\mathbf{D}\, f(x) = 1$ and the derivative of $g(x) = 5$ is $\mathbf{D}\, g(x) = 0$. What are the derivatives of the elementary combinations: $3 \cdot f$, $f + g$, $f - g$, $f \cdot g$ and $\frac{f}{g}$?

Solution. The first three derivatives follow "nice" patterns:

$$\begin{aligned}
\mathbf{D}(3 \cdot f(x)) &= \mathbf{D}(3x) = 3 = 3 \cdot 1 = 3 \cdot \mathbf{D}(f(x)) \\
\mathbf{D}(f(x) + g(x)) &= \mathbf{D}(x+5) = 1 = 1 + 0 = \mathbf{D}(f(x)) + \mathbf{D}(g(x)) \\
\mathbf{D}(f(x) - g(x)) &= \mathbf{D}(x-5) = 1 = 1 - 0\,\mathbf{D}(f(x)) - \mathbf{D}(g(x))
\end{aligned}$$

yet the other two derivatives fail to follow the same "nice" patterns: $\mathbf{D}(f(x) \cdot g(x)) = \mathbf{D}(5x) = 5$ but $\mathbf{D}(f(x)) \cdot \mathbf{D}(g(x)) = 1 \cdot 0 = 0$, and $\mathbf{D}\left(\frac{f(x)}{g(x)}\right) = \mathbf{D}\left(\frac{x}{5}\right) = \frac{1}{5}$ but $\frac{\mathbf{D}(f(x))}{\mathbf{D}(g(x))} = \frac{1}{0}$ is undefined. ◀

The two very simple functions in the previous example show that, in general, $\mathbf{D}(f \cdot g) \neq \mathbf{D}(f) \cdot \mathbf{D}(g)$ and $\mathbf{D}\left(\frac{f}{g}\right) \neq \frac{\mathbf{D}(f)}{\mathbf{D}(g)}$.

Practice 2. For $f(x) = 6x + 8$ and $g(x) = 2$, compute the derivatives of $3 \cdot f$, $f + g$, $f - g$, $f \cdot g$ and $\frac{f}{g}$.

Main Differentiation Theorem:

If f and g are differentiable at x, then:

(a) **Constant Multiple Rule**:

$$\mathbf{D}(k \cdot f(x)) = k \cdot \mathbf{D}(f(x))$$

(b) **Sum Rule**:

$$\mathbf{D}(f(x) + g(x)) = \mathbf{D}(f(x)) + \mathbf{D}(g(x))$$

(c) **Difference Rule**:

$$\mathbf{D}(f(x) - g(x)) = \mathbf{D}(f(x)) - \mathbf{D}(g(x))$$

(d) **Product Rule**:

$$\mathbf{D}(f(x) \cdot g(x)) = f(x) \cdot \mathbf{D}(g(x)) + g(x) \cdot \mathbf{D}(f(x))$$

(e) **Quotient Rule**:

$$\mathbf{D}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \cdot \mathbf{D}(f(x)) - f(x) \cdot \mathbf{D}(g(x))}{[g(x)]^2}$$

This theorem says that the simple patterns in the previous example for constant multiples of functions and sums and differences of functions are true for all differentiable functions. It also includes the correct patterns for derivatives of products and quotients of differentiable functions.

The proofs of parts (a), (b) and (c) of this theorem are straightforward, but parts (d) and (e) require some clever algebraic manipulations. Let's look at some examples before tackling the proof.

Example 5. Recall that $\mathbf{D}(x^2) = 2x$ and $\mathbf{D}(\sin(x)) = \cos(x)$. Find $\mathbf{D}(3\sin(x))$ and $\frac{d}{dx}(5x^2 - 7\sin(x))$.

Solution. Computing $\mathbf{D}(3\sin(x))$ requires part (a) of the theorem with $k = 3$ and $f(x) = \sin(x)$ so $\mathbf{D}(3 \cdot \sin(x)) = 3 \cdot \mathbf{D}(\sin(x)) = 3\cos(x)$, while $\dfrac{d}{dx}(5x^2 - 7\sin(x))$ uses part (c) of the theorem with $f(x) = 5x^2$ and $g(x) = 7\sin(x)$ so:

$$\begin{aligned}\frac{d}{dx}(5x^2 - 7\sin(x)) &= \frac{d}{dx}(5x^2) - \frac{d}{dx}(7\sin(x)) \\ &= 5 \cdot \frac{d}{dx}(x^2) - 7\frac{d}{dx}(\sin(x)) \\ &= 5(2x) - 7(\cos(x))\end{aligned}$$

which simplifies to $10x - 7\cos(x)$. ◀

Practice 3. Find $\mathbf{D}(x^3 - 5\sin(x))$ and $\dfrac{d}{dx}(\sin(x) - 4x^3)$.

Practice 4. The table below gives the values of functions f and g, as well as their derivatives, at various points. Fill in the missing values for $\mathbf{D}(3 \cdot f(x))$, $\mathbf{D}(2 \cdot f(x) + g(x))$ and $\mathbf{D}(3 \cdot g(x) - f(x))$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$\mathbf{D}(3f(x))$	$\mathbf{D}(2f(x)+g(x))$	$\mathbf{D}(3g(x)-f(x))$
0	3	−2	−4	3			
1	2	−1	1	0			
2	4	2	3	1			

Practice 5. Use the Main Differentiation Theorem to complete the table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$\mathbf{D}(f(x) \cdot g(x))$	$\mathbf{D}\left(\frac{f(x)}{g(x)}\right)$	$\mathbf{D}\left(\frac{g(x)}{f(x)}\right)$
0	3	−2	−4	3			
1	2	−1	1	0			
2	4	2	3	1			

Example 6. Determine $\mathbf{D}(x^2 \cdot \sin(x))$ and $\dfrac{d}{dx}\left(\dfrac{x^3}{\sin(x)}\right)$.

Solution. (a) Use the Product Rule with $f(x) = x^2$ and $g(x) = \sin(x)$:

$$\begin{aligned}\mathbf{D}(x^2 \cdot \sin(x)) &= \mathbf{D}(f(x) \cdot g(x)) = f(x) \cdot \mathbf{D}(g(x)) + g(x) \cdot \mathbf{D}(f(x)) \\ &= x^2 \cdot \mathbf{D}(\sin(x)) + \sin(x) \cdot \mathbf{D}(x^2) \\ &= x^2 \cdot \cos(x) + \sin(x) \cdot 2x = x^2\cos(x) + 2x\sin(x)\end{aligned}$$

Many calculus students find it easier to remember the product rule in words: "the first function times the derivative of the second plus the second function times the derivative of the first."

(b) Use the Quotient Rule with $f(x) = x^3$ and $g(x) = \sin(x)$:

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^3}{\sin(x)}\right) &= \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\\ &= \frac{g(x)\cdot\mathbf{D}(f(x)) - f(x)\cdot\mathbf{D}(g(x))}{[g(x)]^2}\\ &= \frac{\sin(x)\cdot\mathbf{D}(x^3) - x^3\cdot\mathbf{D}(\sin(x))}{[\sin(x)]^2}\\ &= \frac{\sin(x)\cdot 3x^2 - x^3\cdot\cos(x)}{\sin^2(x)}\\ &= \frac{3x^2\sin(x) - x^3\cdot\cos(x)}{\sin^2(x)}\end{aligned}$$

The quotient rule in words: "the bottom time the derivative of the top minus the top times the derivative of the bottom, all over the bottom squared."

which could also be rewritten in terms of $\csc(x)$ and $\cot(x)$. ◀

Practice 6. Determine $\mathbf{D}((x^2+1)(7x-3))$, $\frac{d}{dt}\left(\frac{3t-2}{5t+1}\right)$ and $\mathbf{D}\left(\frac{\cos(x)}{x}\right)$.

Now that we've seen how to use the theorem, let's prove it.

Proof. The only general fact we have about derivatives is the definition as a limit, so our proofs here will need to recast derivatives as limits and then use some results about limits. The proofs involve applications of the definition of the derivative and results about limits.

(a) Using the derivative definition and the limit laws:

$$\begin{aligned}\mathbf{D}(k\cdot f(x)) &= \lim_{h\to 0}\frac{k\cdot f(x+h) - k\cdot f(x)}{h}\\ &= \lim_{h\to 0} k\cdot\frac{f(x+h)-f(x)}{h}\\ &= k\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = k\cdot\mathbf{D}(f(x))\end{aligned}$$

(b) You try it (see Practice problem that follows).

(c) Once again using the derivative definition and the limit laws:

$$\begin{aligned}\mathbf{D}(f(x)-g(x)) &= \lim_{h\to 0}\frac{[f(x+h)-g(x+h)]-[f(x)-g(x)]}{h}\\ &= \lim_{h\to 0}\frac{[f(x+h)-f(x)]-[g(x+h)-g(x)]}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} - \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= \mathbf{D}(f(x)) - \mathbf{D}(g(x))\end{aligned}$$

The proofs of parts (d) and (e) of the theorem are more complicated but only involve elementary techniques, used in just the right way.

Sometimes we will omit such computational proofs, but the Product and Quotient Rules are fundamental techniques you will need hundreds of times.

(d) By the hypothesis, f and g are differentiable, so:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = f'(x)$$

and:

$$\lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = g'(x)$$

Also, both f and g are continuous (why?) so $\lim_{h\to 0} f(x+h) = f(x)$ and $\lim_{h\to 0} g(x+h) = g(x)$.

Let $P(x) = f(x) \cdot g(x)$. Then $P(x+h) = f(x+h) \cdot g(x+h)$ and:

$$\begin{aligned} \mathbf{D}(f(x) \cdot g(x)) = \mathbf{D}(P(x)) &= \lim_{h\to 0} \frac{P(x+h) - P(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h} \end{aligned}$$

At this stage we need to use some cleverness to add and subtract $f(x) \cdot g(x+h)$ from the numerator (you'll see why shortly):

$$\lim_{h\to 0} \frac{f(x+h) \cdot g(x+h) + [-f(x) \cdot g(x+h) + f(x) \cdot g(x+h)] - f(x)g(x)}{h}$$

We can then split this giant fraction into two more manageable limits:

$$\lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h} + \lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}$$

and then factor out a common factor from each numerator:

$$\lim_{h\to 0} g(x+h) \cdot \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} f(x) \cdot \frac{g(x+h) - g(x)}{h}$$

Taking limits of each piece (and using the continuity of $g(x)$) we get:

$$\mathbf{D}(f(x) \cdot g(x)) = g(x) \cdot f'(x) + f(x) \cdot g'(x) = g \cdot \mathbf{D} f + f \cdot \mathbf{D} g$$

The steps for a proof of the Quotient Rule appear in Problem 55. ☐

Practice 7. Prove the Sum Rule: $\mathbf{D}(f(x) + g(x)) = \mathbf{D}(f(x)) + \mathbf{D}(g(x))$. (Refer to the proof of part (c) for guidance.)

Using the Differentiation Rules

You definitely need to memorize the differentiation rules, but it is vitally important that you also know **how** to use them. Sometimes it is clear that the function we want to differentiate is a sum or product of two obvious functions, but we commonly need to differentiate functions that involve several operations and functions. Memorizing the differentiation rules is only the first step in learning to use them.

Example 7. Calculate $\mathbf{D}(x^5 + x \cdot \sin(x))$.

Solution. This function is more difficult because it involves both an addition and a multiplication. Which rule(s) should we use—or, more importantly, which rule should we use **first**?

First apply the Sum Rule to trade one derivative for two easier ones:

$$\begin{aligned}\mathbf{D}(x^5 + x \cdot \sin(x)) &= \mathbf{D}(x^5) + \mathbf{D}(x \cdot \sin(x)) \\ &= 5x^4 + [x \cdot \mathbf{D}(\sin(x)) + \sin(x) \cdot \mathbf{D}(x)] \\ &= 5x^4 + x \cdot \cos(x) + \sin(x)\end{aligned}$$

This last expression involves no more derivatives, so we are done. ◀

If instead of computing the derivative you were evaluating the function $x^5 + x\sin(x)$ for some particular value of x, you would:

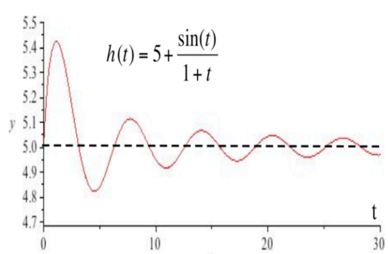
- raise x to the 5th power
- calculate $\sin(x)$
- multiply $\sin(x)$ by x and, finally,
- **add** (sum) the values of x^5 and $x\sin(x)$

Notice that the **final** step of your **evaluation** of f indicates the **first** rule to use to calculate the **derivative** of f.

Practice 8. Which differentiation rule should you apply **first** for each of the following?

(a) $x \cdot \cos(x) - x^3 \cdot \sin(x)$ (b) $(2x-3)\cos(x)$

(a) $2\cos(x) - 7x^2$ (b) $\dfrac{\cos(x)+3x}{\sqrt{x}}$



Practice 9. Calculate $D\left(\dfrac{x^2-5}{\sin(x)}\right)$ and $\dfrac{d}{dt}\left(\dfrac{t^2-5}{t \cdot \sin(t)}\right)$.

Example 8. A mass attached to a spring is oscillating up and down. Over time, the motion becomes "damped" because of friction and air resistance, and the height (in feet) of the mass after t seconds is given by $h(t) = 5 + \dfrac{\sin(t)}{1+t}$. What are the height and velocity of the weight after 2 seconds?

Solution. The height is $h(2) = 5 + \dfrac{\sin(2)}{1+2} \approx 5 + \dfrac{0.909}{3} = 5.303$ feet above the ground. The velocity is $h'(2)$, so we must first compute $h'(t)$ and then evaluate the derivative at time $t = 2$.

$$\begin{aligned}h'(t) &= 0 + \frac{(1+t)\cdot \mathbf{D}(\sin(t)) - \sin(t)\cdot \mathbf{D}(1+t)}{(1+t)^2} \\ &= \frac{(1+t)\cdot\cos(t) - \sin(t)}{(1+t)^2}\end{aligned}$$

so $h'(2) = \dfrac{3\cos(2) - \sin(2)}{9} \approx \dfrac{-2.158}{9} \approx -0.24$ feet per second. ◀

Practice 10. What are the height and velocity of the weight in the previous example after 5 seconds? What are the height and velocity of the weight be after a "long time" has passed?

Example 9. Calculate $\mathbf{D}(x \cdot \sin(x) \cdot \cos(x))$.

Solution. Clearly we need to use the Product Rule, because the only operation in this function is multiplication. But the Product Rule deals with a product of **two** functions and here we have the product of three: x and $\sin(x)$ and $\cos(x)$. If, however, we think of our two functions as $f(x) = x \cdot \sin(x)$ and $g(x) = \cos(x)$, then we do have the product of two functions and:

$$\begin{aligned}
\mathbf{D}(x \cdot \sin(x) \cdot \cos(x)) &= \mathbf{D}(f(x) \cdot g(x)) \\
&= f(x) \cdot \mathbf{D}(g(x)) + g(x) \cdot \mathbf{D}(f(x)) \\
&= x\sin(x) \cdot \mathbf{D}(\cos(x)) + \cos(x) \cdot \mathbf{D}(x\sin(x))
\end{aligned}$$

We are not done, but we have traded one hard derivative for two easier ones. We know that $\mathbf{D}(\cos(x)) = -\sin(x)$ and we can use the Product Rule (again) to calculate $\mathbf{D}(x\sin(x))$. Then the last line of our calculation above becomes:

$$x\sin(x) \cdot [-\sin(x)] + \cos(x) \cdot [x\,\mathbf{D}(\sin(x)) + \sin(x)\,\mathbf{D}(x)]$$

and then:

$$-x\sin^2(x) + \cos(x)\,[x\cos(x) + \sin(x)(1)]$$

which simplifies to $-x\sin^2(x) + x\cos^2(x) + \cos(x)\sin(x)$. ◀

Evaluating a Derivative at a Point

The derivative of a function $f(x)$ is a new function $f'(x)$ that tells us the slope of the line tangent to the graph of f at each point x. To find the slope of the tangent line at a particular point $(c, f(c))$ on the graph of f, we should first calculate the derivative $f'(x)$ and then evaluate the function $f'(x)$ at the point $x = c$ to get the number $f'(c)$. If you mistakenly evaluate f first, you get a number $f(c)$, and the derivative of a constant is always equal to 0.

Example 10. Determine the slope of the line tangent to the graph of $f(x) = 3x + \sin(x)$ at $(0, f(0))$ and $(1, f(1))$.

Solution. $f'(x) = \mathbf{D}(3x + \sin(x)) = \mathbf{D}(3x) + \mathbf{D}(\sin(x)) = 3 + \cos(x)$. When $x = 0$, the graph of $y = 3x + \sin(x)$ goes through the point $(0, 3(0) + \sin(0)) = (0, 0)$ with slope $f'(0) = 3 + \cos(0) = 4$. When $x = 1$, the graph goes through the point $(1, 3(1) + \sin(1)) \approx (1, 3.84)$ with slope $f'(1) = 3 + \cos(1) \approx 3.54$. ◀

Practice 11. Where do $f(x) = x^2 - 10x + 3$ and $g(x) = x^3 - 12x$ have horizontal tangent lines?

Important Information and Results

This section, like the last one, contains a great deal of important information that we will continue to use throughout the rest of the course, so we collect here some of those important results.

Differentiability and Continuity: If a function is differentiable then it must be continuous. If a function is not continuous then it cannot be differentiable. A function may be continuous at a point and not differentiable there.

Graphically: *Continuous* means "connected"; *differentiable* means "continuous, smooth and not vertical."

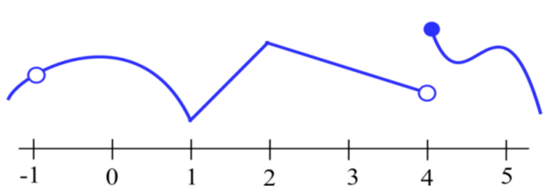
Differentiation Patterns:

- $[k \cdot f(x)]' = k \cdot f'(x)$
- $[f(x) + g(x)]' = f'(x) + g'(x)$
- $[f(x) - g(x)]' = f'(x) - g'(x)$
- $[f(x) \cdot g(x)]' = f(x) \cdot g'(x) + g(x) \cdot f'(x)$
- $\left[\dfrac{f(x)}{g(x)}\right]' = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$
- The *final step* used to evaluate a function f indicates the *first rule* used to differentiate f.

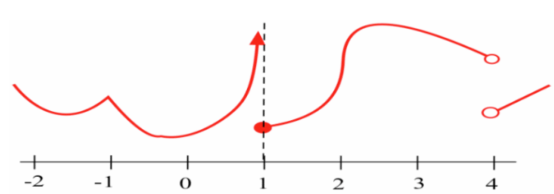
Evaluating a derivative at a point: First differentiate and *then* evaluate.

2.2 *Problems*

1. Use the graph of $y = f(x)$ below to determine:
   (a) at which integers f is continuous.
   (b) at which integers f is differentiable.



2. Use the graph of $y = g(x)$ below to determine:
   (a) at which integers g is continuous.
   (b) at which integers g is differentiable.



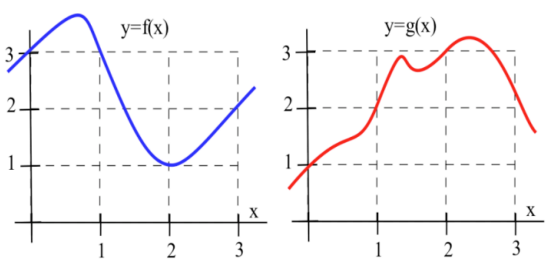
3. Use the values given in the table to determine the values of $f \cdot g$, $\mathbf{D}(f \cdot g)$, $\frac{f}{g}$ and $\mathbf{D}(\frac{f}{g})$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$f(x) \cdot g(x)$	$\mathbf{D}(f(x) \cdot g(x))$	$\frac{f(x)}{g(x)}$	$\mathbf{D}(\frac{f(x)}{g(x)})$
0	2	3	5	5				
1	−3	2	5	−2				
2	0	−3	2	4				
3	1	−1	0	3				

4. Use the values given in the table to determine the values of $f \cdot g$, $\mathbf{D}(f \cdot g)$, $\frac{f}{g}$ and $\mathbf{D}(\frac{f}{g})$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$f(x) \cdot g(x)$	$\mathbf{D}(f(x) \cdot g(x))$	$\frac{f(x)}{g(x)}$	$\mathbf{D}(\frac{f(x)}{g(x)})$
0	4	2	3	−3				
1	0	3	2	1				
2	−2	5	0	−1				
3	−1	−2	−3	4				

5. Use the information in the figure below to plot the values of the functions $f+g$, $f \cdot g$ and $\frac{f}{g}$ and their derivatives at $x = 1$, 2 and 3.



6. Use the information in the figure above to plot the values of the functions $2f$, $f-g$ and $\frac{g}{f}$ and their derivatives at $x = 1$, 2 and 3.

7. Calculate $\mathbf{D}((x-5)(3x+7))$ by:

(a) using the product rule.

(b) expanding and then differentiating.

Verify that both methods give the same result.

In Problems 8-12, compute each derivative.

8. $D(x \cdot \sin(x))$

9. $\frac{d}{dx}\left(\frac{\cos(x)}{x^2}\right)$

10. $\mathbf{D}(\sin(x) + \cos(x))$

11. (a) $\mathbf{D}(\sin^2(x))$ (b) $\mathbf{D}(\cos^2(x))$

12. (a) $\mathbf{D}(\sin(x))$

(b) $\frac{d}{dx}(\sin(x) + 7)$

(c) $\mathbf{D}(\sin(x) - 8000)$ and $\mathbf{D}(\sin(x) + k)$

13. Find values for the constants a, b and c so that the parabola $f(x) = ax^2 + bx + c$ has $f(0) = 0$, $f'(0) = 0$ and $f'(10) = 30$.

14. If f is a differentiable function, how are the:

(a) graphs of $y = f(x)$ and $y = f(x) + k$ related?

(b) derivatives of $f(x)$ and $f(x) + k$ related?

15. If f and g are differentiable functions that always differ by a constant ($f(x) - g(x) = k$ for all x) then what can you conclude about their graphs? Their derivatives?

16. If f and g are differentiable functions whose sum is a constant ($f(x) + g(x) = k$ for all x) then what can you conclude about their graphs? Their derivatives?

17. If the product of f and g is a constant (that is, $f(x) \cdot g(x) = k$ for all x) then how are $\frac{\mathbf{D}(f(x))}{f(x)}$ and $\frac{\mathbf{D}(g(x))}{g(x)}$ related?

18. If the quotient of f and g is a constant ($\frac{f(x)}{g(x)} = k$ for all x) then how are $g \cdot f'$ and $f \cdot g'$ related?

In Problems 19–28:

(a) calculate $f'(1)$

(b) determine where $f'(x) = 0$.

19. $f(x) = x^2 - 5x + 13$
20. $f(x) = 5x^2 - 40x + 73$
21. $f(x) = 3x - 2\cos(x)$
22. $f(x) = |x + 2|$
23. $f(x) = x^3 + 9x^2 + 6$
24. $f(x) = x^3 + 3x^2 + 3x - 1$
25. $f(x) = x^3 + 2x^2 + 2x - 1$
26. $f(x) = \dfrac{7x}{x^2 + 4}$
27. $f(x) = x \cdot \sin(x)$ and $0 \le x \le 5$. (You may need to use the Bisection Algorithm or the "trace" option on a calculator to approximate where $f'(x) = 0$.)
28. $f(x) = Ax^2 + Bx + CA$, where B and C are constants and $A \neq 0$ is constant.
29. $f(x) = x^3 + Ax^2 + Bx + C$ with constants A, B and C. Can you find conditions on the constants A, B and C that will guarantee that the graph of $y = f(x)$ has two distinct "turning points?" (Here a "turning point" means a place where the curve changes from increasing to decreasing or from decreasing to increasing, like vertex of a parabola.)

Where are the functions in differentiable?

30. $f(x) = |x|\cos(x)$
31. $f(x) = \dfrac{x-5}{x+3}$
32. $f(x) = \tan(x)$
33. $f(x) = \dfrac{x^2 + x}{x^2 - 3x}$
34. $f(x) = |x^2 - 4|$
35. $f(x) = |x^3 - 1|$
36. $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sin(x) & \text{if } x \ge 0 \end{cases}$
37. $f(x) = \begin{cases} x & \text{if } x < 0 \\ \sin(x) & \text{if } x \ge 0 \end{cases}$
38. For what value(s) of A is

$$f(x) = \begin{cases} Ax - 4 & \text{if } x < 2 \\ x^2 + x & \text{if } x \ge 2 \end{cases}$$

differentiable at $x = 2$?

39. For what values of A and B is

$$f(x) = \begin{cases} Ax + B & \text{if } x < 1 \\ x^2 + x & \text{if } x \ge 1 \end{cases}$$

differentiable at $x = 1$?

40. An arrow shot straight up from ground level (get out of the way!) with an initial velocity of 128 feet per second will be at height $h(x) = -16x^2 + 128x$ feet after x seconds (see figure below).

(a) Determine the velocity of the arrow when $x = 0$, 1 and 2 seconds.

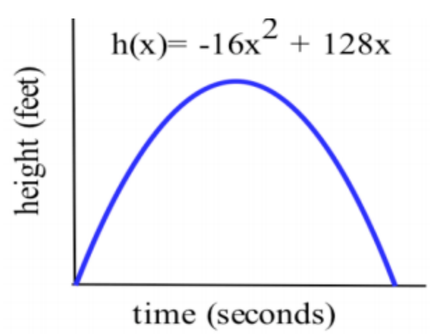
(b) What is the velocity of the arrow, $v(x)$, at any time x?

(c) At what time x will the velocity of the arrow be 0?

(d) What is the greatest height the arrow reaches?

(e) How long will the arrow be aloft?

(f) Use the answer for the velocity in part (b) to determine the acceleration, $a(x) = v'(x)$, at any time x.



41. If an arrow is shot straight up from ground level on the moon with an initial velocity of 128 feet per second, its height will be $h(x) = -2.65x^2 + 128x$ feet after x seconds. Redo parts (a)–(e) of problem 40 using this new formula for $h(x)$.

42. In general, if an arrow is shot straight upward with an initial velocity of 128 feet per second from ground level on a planet with a constant gravitational acceleration of g feet per second^2 then its height will be $h(x) = -\frac{g}{2}x^2 + 128x$ feet after x seconds. Answer the questions in problem 40 for arrows shot on Mars and Jupiter.

object	g (ft/sec^2)	g (cm/sec^2)
Mercury	11.8	358
Venus	20.1	887
Earth	32.2	981
moon	5.3	162
Mars	12.3	374
Jupiter	85.3	2601
Saturn	36.6	1117
Uranus	34.4	1049
Neptune	43.5	1325

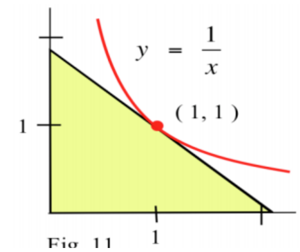
Source: *CRC Handbook of Chemistry and Physics*

43. If an object on Earth is propelled upward from ground level with an initial velocity of v_0 feet per second, then its height after x seconds will be $h(x) = -16x^2 + v_0x$.
   (a) Find the object's velocity after x seconds.
   (b) Find the greatest height the object will reach.
   (c) How long will the object remain aloft?

44. In order for a 6-foot-tall basketball player to dunk the ball, the player must achieve a vertical jump of about 3 feet. Use the information in the previous problems to answer the following questions.
   (a) What is the smallest initial vertical velocity the player can have and still dunk the ball?
   (b) With the initial velocity achieved in part (a), how high would the player jump on the moon?

45. The best high jumpers in the world manage to lift their centers of mass approximately 6.5 feet above the ground.
   (a) What is the initial vertical velocity these high jumpers attain?
   (b) How long are these high jumpers in the air?
   (c) How high would they lift their centers of mass on the moon?

46. (a) Find an equation for the line L that is tangent to the curve $y = \dfrac{1}{x}$ at the point $(1, 1)$.
   (b) Determine where L intersects the x-axis and the y-axis.
   (c) Determine the area of the region in the first quadrant bounded by L, the x-axis and the y-axis (see figure below).



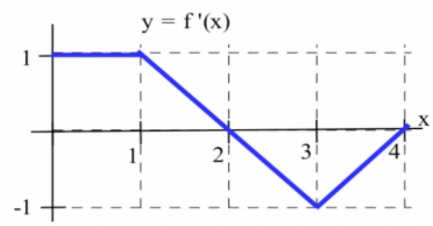
47. (a) Find an equation for the line L that is tangent to the curve $y = \dfrac{1}{x}$ at the point $(2, \frac{1}{2})$.
   (b) Graph $y = \frac{1}{x}$ and L and determine where L intersects the x-axis and the y-axis.
   (c) Determine the area of the region in the first quadrant bounded by L, the x-axis and the y-axis.

48. (a) Find an equation for the line L that is tangent to the curve $y = \dfrac{1}{x}$ at the point $(p, \frac{1}{p})$ (assuming $p \neq 0$).
   (b) Determine where L intersects the x-axis and the y-axis.
   (c) Determine the area of the region in the first quadrant bounded by L, the x-axis and the y-axis.
   (d) How does the area of the triangle in part (c) depend on the initial point $(p, \frac{1}{p})$?

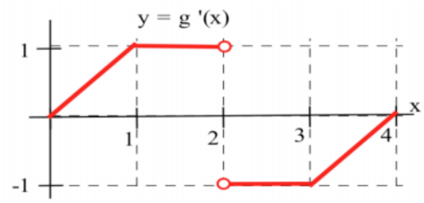
49. Find values for the coefficients a, b and c so that the parabola $f(x) = ax^2 + bx + c$ goes through the point $(1, 4)$ and is tangent to the line $y = 9x - 13$ at the point $(3, 14)$.

50. Find values for the coefficients a, b and c so that the parabola $f(x) = ax^2 + bx + c$ goes through the point $(0, 1)$ and is tangent to the line $y = 3x - 2$ at the point $(2, 4)$.

51. (a) Find a function f so that $\mathbf{D}(f(x)) = 3x^2$.
    (b) Find another function g with $\mathbf{D}(g(x)) = 3x^2$.
    (c) Can you find more functions whose derivatives are $3x^2$?
52. (a) Find a function f so that $f'(x) = 6x + \cos(x)$.
    (b) Find another function g with $g'(x) = f'(x)$.
53. The graph of $y = f'(x)$ appears below.
    (a) Assume $f(0) = 0$ and sketch a graph of $y = f(x)$.
    (b) Assume $f(0) = 1$ and graph $y = f(x)$.



54. The graph of $y = g'(x)$ appears below. Assume that g is continuous.
    (a) Assume $g(0) = 0$ and sketch a graph of $y = g(x)$.
    (b) Assume $g(0) = 1$ and graph $y = g(x)$.



55. Assume that f and g are differentiable functions and that $g(x) \neq 0$. State why each step in the following proof of the Quotient Rule is valid.

Proof of the Quotient Rule

$$\mathbf{D}\left(\frac{f(x)}{g(x)}\right) = \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{f(x+h)g(x) - g(x+h)f(x)}{g(x+h)g(x)}\right]$$
$$= \lim_{h\to 0} \frac{1}{g(x+h)g(x)}\left[\frac{f(x+h)g(x) + (-f(x)g(x) + f(x)g(x)) - g(x+h)f(x)}{h}\right]$$
$$= \lim_{h\to 0} \frac{1}{g(x+h)g(x)}\left[g(x)\frac{f(x+h) - f(x)}{h} + f(x)\frac{g(x) - g(x+h)}{h}\right]$$
$$= \frac{1}{[g(x)]^2}\left[g(x)\cdot f'(x) - f(x)\cdot g'(x)\right]$$
$$= \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

Practice Answers

1. f is continuous at $x = -1, 0, 2, 4, 6$ and 7.
   f is differentiable at $x = -1, 2, 4,$ and 7.

2. $f(x) = 6x + 8$ and $g(x) = 2$ so $\mathbf{D}(f(x)) = 6$ and $\mathbf{D}(g(x)) = 0$.
   $\mathbf{D}(3\cdot f(x)) = 3\cdot \mathbf{D}(f(x)) = 3(6) = 18$
   $\mathbf{D}(f(x) + g(x)) = \mathbf{D}(f(x)) + \mathbf{D}(g(x)) = 6 + 0 = 6$
   $\mathbf{D}(f(x) - g(x)) = \mathbf{D}(f(x)) - \mathbf{D}(g(x)) = 6 - 0 = 6$
   $\mathbf{D}(f(x)\cdot g(x)) = f(x)g'(x) + g(x)f'(x) = (6x+8)(0) + (2)(6) = 12$
   $\mathbf{D}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} = \frac{(2)(6)-(6x+8)(0)}{2^2} = \frac{12}{4} = 3$

3. $\mathbf{D}(x^3 - 5\sin(x)) = \mathbf{D}(x^3) - 5 \cdot \mathbf{D}(\sin(x)) = 3x^2 - 5\cos(x)$
$\frac{d}{dx}\left(\sin(x) - 4x^3\right) = \frac{d}{dx}\sin(x) - 4 \cdot \frac{d}{dx}x^3 = \cos(x) - 12x^2$

4.

$\mathbf{D}(3f(x))$	$\mathbf{D}(2f(x)+g(x))$	$\mathbf{D}(3g(x)-f(x))$
-6	-1	11
-3	-2	1
6	5	1

5.

$\mathbf{D}(f(x) \cdot g(x))$	$\mathbf{D}\left(\frac{f(x)}{g(x)}\right)$	$\mathbf{D}\left(\frac{g(x)}{f(x)}\right)$
$3 \cdot 3 + (-4)(-2) = 17$	$\frac{-4(-2)-(3)(3)}{(-4)^2} = -\frac{1}{16}$	$\frac{(3)(3)-(-4)(-2)}{3^2} = \frac{1}{9}$
$2 \cdot 0 + 1(-1) = -1$	$\frac{1(-1)-(2)(0)}{1^2} = -1$	$\frac{2(0)-1(-1)}{2^2} = \frac{1}{4}$
$4 \cdot 1 + 3 \cdot 2 = 10$	$\frac{3(2)-(4)(1)}{3^2} = \frac{2}{9}$	$\frac{4(1)-3(2)}{4^2} = -\frac{1}{8}$

6. $\mathbf{D}((x^2+1)(7x-3)) = (x^2+1)\,\mathbf{D}(7x-3) + (7x-3)\,\mathbf{D}(x^2+1)$
$= (x^2+1)(7) + (7x-3)(2x) = 21x^2 - 6x + 7$
or: $\mathbf{D}((x2+1)(7x-3)) = \mathbf{D}(7x^3 - 3x^2 + 7x) = 21x^2 - 6x + 7$
$\frac{d}{dt}\left(\frac{3t-2}{5t+1}\right) = \frac{(5t+1)\,\mathbf{D}(3t-2) - (3t-2)\,\mathbf{D}(5t+1)}{(5t+1)^2} = \frac{(5t+1)(3) - (3t-2)(5)}{(5t+1)^2} = \frac{13}{(5t+1)^2}$
$\mathbf{D}\left(\frac{\cos(x)}{x}\right) = \frac{x\,\mathbf{D}(\cos(x)) - \cos(x)\,\mathbf{D}(x)}{x^2} = \frac{x(-\sin(x)) - \cos(x)(1)}{x^2} = \frac{-x \cdot \sin(x) - \cos(x)}{x^2}$

7. Mimicking the proof of the Difference Rule:

$$\begin{aligned}
\mathbf{D}(f(x)+g(x)) &= \lim_{h \to 0} \frac{[f(x+h)+g(x+h)] - [f(x)+g(x)]}{h} \\
&= \lim_{h \to 0} \frac{[f(x+h)-f(x)] + [g(x+h)-g(x)]}{h} \\
&= \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h \to 0} \frac{g(x+h)-g(x)}{h} \\
&= \mathbf{D}(f(x)) + \mathbf{D}(g(x))
\end{aligned}$$

8. (a) difference rule (b) product rule (c) difference rule (d) quotient rule

9. $\mathbf{D}\left(x^2 - 5\sin(x)\right) = \frac{\sin(x)\,\mathbf{D}(x^2-5) - (x^2-5)\,\mathbf{D}(\sin(x))}{(\sin(x))^2} = \frac{\sin(x)(2x) - (x^2-5)\cos(x)}{\sin^2(x)}$
$\frac{d}{dt}\left(t^2 - 5t \cdot \sin(t)\right) = \frac{t \cdot \sin(t)\,\mathbf{D}(t^2-5) - (t^2-5)\,\mathbf{D}(t \cdot \sin(t))}{(t \cdot \sin(t))^2} = \frac{t \cdot \sin(t)(2t) - (t^2-5)\,[t\cos(t) + \sin(t)]}{t^2 \cdot \sin^2(t)}$

10. $h(5) = 5 + \frac{\sin(5)}{1+5} \approx 4.84$ ft.; $v(5) = h'(5) = \frac{(1+5)\cos(5) - \sin(5)}{(1+5)^2} \approx 0.074$ ft/sec.
"long time": $h(t) = 5 + \frac{\sin(t)}{1+t} \approx 5$ feet when t is very large;
$h'(t) = \frac{(1+t)\cos(t) - \sin(t)}{(1+t)^2} = \frac{\cos(t)}{1+t} - \frac{\sin(t)}{(1+t)^2} \approx 0$ ft/sec when t is very large.

11. $f'(x) = 2x - 10$ so $f'(x) \Longrightarrow 2x - 10 = 0 \Rightarrow x = 5$.
$g'(x) = 3x^2 - 12$ so $g'(x) = 0 \Rightarrow 3x^2 - 12 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$.