

2.1.1 Introduction to Fractions and Mixed Numbers

Learning Objective(s)

- 1 Identify the numerator and denominator of a fraction.
- 2 Represent a fraction as part of a whole or part of a set.

Introduction

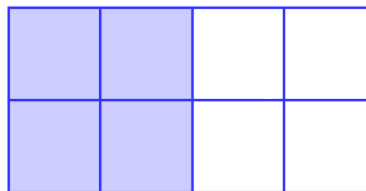
Many problems in mathematics deal with whole numbers, which are used to count whole units of things. For example, you can count students in a classroom and the number of dollar bills. You need other kinds of numbers to describe units that are not whole. For example, an aquarium might be partly full. A group may have a meeting, but only some of the members are present.

Fractions are numbers used to refer to a part of a whole. This includes measurements that cannot be written as whole numbers. For example, the width of a piece of notebook paper is more than 8 inches but less than 9 inches. The part longer than 8 inches is written as a fraction. Here, you will investigate how fractions can be written and used to represent quantities that are parts of the whole.

Identifying Numerators and Denominators

Objective 1

A whole can be divided into parts of equal size. In the example below, a rectangle has been divided into eight equal squares. Four of these eight squares are shaded.



The shaded area can be represented by a fraction. A fraction is written vertically as two numbers with a line between them.

The **denominator** (the bottom number) represents the number of equal parts that make up the whole. The **numerator** (the top number) describes the number of parts that you are describing. So returning to the example above, the rectangle has been divided into 8 equal parts, and 4 of them have been shaded. You can use the fraction $\frac{4}{8}$ to describe the shaded part of the whole.

$\frac{4}{8}$ ← The numerator tells how many parts are shaded.

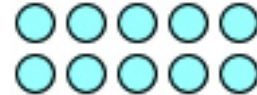
$\frac{4}{8}$ ← The denominator tells how many parts are required to make up the whole.



The rectangle model above provides a good, basic introduction to fractions. However, what do you do with situations that cannot be as easily modeled by shading part of a figure? For example, think about the following situation:

Marc works as a Quality Assurance Manager at an automotive plant. Every hour he inspects 10 cars; $\frac{4}{5}$ of those pass inspection.

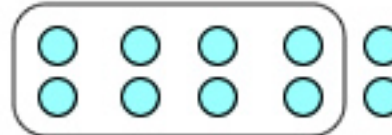
In this case, 10 cars make up the whole group. Each car can be represented as a circle, as shown below.



To show $\frac{4}{5}$ of the whole group, you first need to divide the whole group into 5 equal parts. (You know this because the fraction has a denominator of 5.)

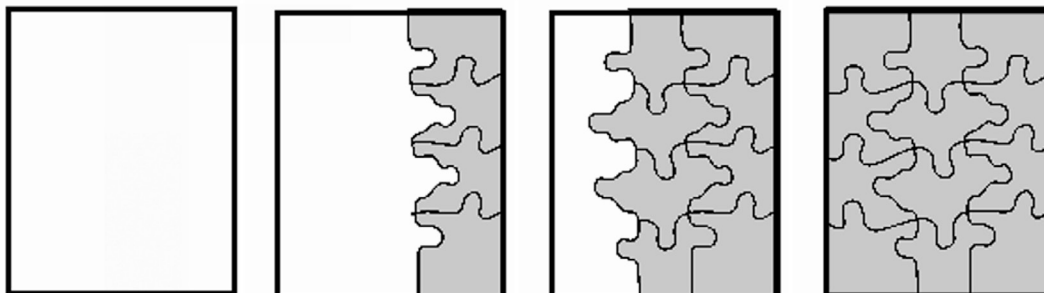


To show $\frac{4}{5}$, circle 4 of the equal parts.



Here is another example. Imagine that Aneesh is putting together a puzzle made of 12 pieces. At the beginning, none of the pieces have been put into the puzzle. This means that $\frac{0}{12}$ of the puzzle is complete. Aneesh then puts four pieces together. The puzzle is $\frac{4}{12}$ complete. Soon, he adds four more pieces. Eight out of twelve pieces are now connected. This fraction can be written as $\frac{8}{12}$. Finally, Aneesh adds four more pieces.

The puzzle is whole, using all 12 pieces. The fraction can be written as $\frac{12}{12}$.



$\frac{0}{12}$

$\frac{4}{12}$

$\frac{8}{12}$

$\frac{12}{12}$

Note that the number in the denominator cannot be zero. The denominator tells how many parts make up the whole. So if this number is 0, then there are no parts and therefore there can be no whole.

The numerator can be zero, as it tells how many parts you are describing. Notice that in the puzzle example above, you can use the fraction $\frac{0}{12}$ to represent the state of the puzzle when 0 pieces have been placed.

Fractions can also be used to analyze data. In the data table below, 3 out of 5 tosses of a coin came up heads, and 2 out of five tosses came up tails. Out of the total number of coin tosses, the portion that was heads can be written as $\frac{3}{5}$. The portion that was tails can be written as $\frac{2}{5}$.

Coin Toss	Result
1	Heads
2	Tails
3	Heads
4	Heads
5	Tails

Self Check A

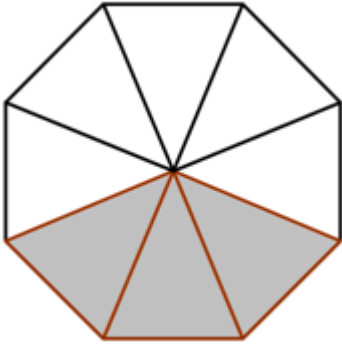
Sophia, Daphne, and Charlie are all participating in a relay race to raise money for charity. First, Sophia will run 2 miles. Then, Daphne will run 5 miles. Finally, Charlie will end the race by running 3 miles. What fraction of the race will Daphne run?

Parts of a Whole

The “parts of a whole” concept can be modeled with pizzas and pizza slices. For example, imagine a pizza is cut into 4 pieces, and someone takes 1 piece. Now, $\frac{1}{4}$ of the pizza is gone and $\frac{3}{4}$ remains. Note that both of these fractions have a denominator of 4, which refers to the number of slices the whole pizza has been cut into.



Example	
Problem	Joaquim bakes a blueberry pie for a potluck dinner. The total pie is cut into 6 equal slices. After everybody eats dessert, only one slice of the pie remains. What fraction of the pie remains?
	$\frac{?}{6}$ The pie was cut into six equal slices, so the denominator of the fraction will be 6. $\frac{1}{6}$ Only 1 slice remains, so the numerator of the fraction will be 1.
<i>Answer</i>	$\frac{1}{6}$ of the pie remains.

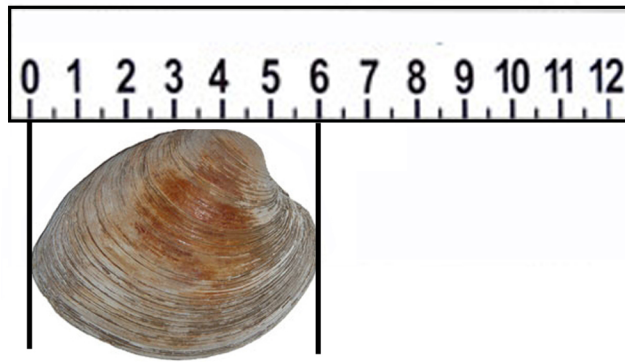
Example	
Problem	 <p>Write a fraction to represent the portion of the octagon that is <i>not</i> shaded.</p>
	$\frac{?}{8}$ The octagon has eight equal sections, so the denominator of the fraction will be 8. $\frac{5}{8}$ Five sections are not shaded, so the numerator of the fraction will be 5.
<i>Answer</i>	$\frac{5}{8}$ of the octagon is not shaded.

Measurement Contexts

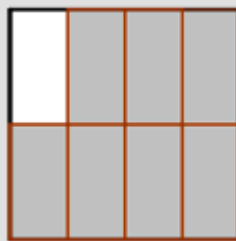
You can use a fraction to represent the quantity in a container. This measuring cup is $\frac{3}{4}$ filled with a liquid. Note that if the cup were $\frac{4}{4}$ full, it would be a whole cup.



You can also use fractions in measuring the length, width, or height of something that is not a full unit. Using a 12-inch ruler, you measure a shell that is 6 inches long. You know that 12 inches equals one foot. So, the length of this shell is $\frac{6}{12}$ of a foot; the 12-inch ruler is the “whole”, and the length of the shell is the “part.”



Self Check B



Which fraction represents the portion of the shape that is shaded?

Summary

Fractions are used to represent parts of a whole. You can use fractions when describing substances, quantities, or diagrams that are not complete. You also use fractions to describe numbers of people or objects that do not make up a complete group. Fractions are written with a numerator and denominator. The numerator (above the fraction bar) tells the number of parts being described, and the denominator (below the fraction bar) tells the number of parts that make up the whole.

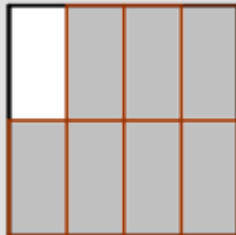
2.1.1 Self Check Solutions

Self Check A

Sophia, Daphne, and Charlie are all participating in a relay race to raise money for charity. First, Sophia will run 2 miles. Then, Daphne will run 5 miles. Finally, Charlie will end the race by running 3 miles. What fraction of the race will Daphne run?

The entire race is 10 miles long, and Daphne will run 5 miles. This means she will run $\frac{5}{10}$ of the race.

Self Check B



Which fraction represents the portion of the shape that is shaded?

$$\frac{7}{8}$$

The total number of parts that make up the whole, 8, is the denominator (below the fraction bar). The number of parts that are shaded, 7, is the numerator (above the fraction bar).

2.1.2 Proper and Improper Fractions

Learning Objective(s)

- 1 Identify proper and improper fractions.
- 2 Change improper fractions to mixed numbers.
- 3 Change mixed numbers to improper fractions.

Introduction

Mathematicians use three categories to describe fractions: proper, improper, and mixed.

Fractions that are greater than 0 but less than 1 are called **proper fractions**. In proper fractions, the **numerator** is less than the **denominator**. When a fraction has a numerator that is greater than or equal to the denominator, the fraction is an **improper fraction**. An improper fraction is always 1 or greater than 1. And, finally, a **mixed number** is a combination of a whole number and a proper fraction.

Identify Proper and Improper Fractions

Objective 1

In a proper fraction, the numerator is always less than the denominator. Examples of proper fractions include $\frac{1}{2}$, $\frac{9}{13}$ and $\frac{1}{1001}$.

In an improper fraction, the numerator is always greater than or equal to the denominator. Examples of improper fractions include $\frac{5}{2}$, $\frac{9}{9}$ and $\frac{25}{20}$.

Self Check A

Identify $\frac{5}{3}$ as a proper or improper fraction.

Changing Improper Fractions to Mixed Numbers

Objective 2

An improper fraction can also be written as a **mixed number**. Mixed numbers contain both a whole number and a proper fraction. Examples of mixed numbers include

$8\frac{1}{10}$, $1\frac{19}{20}$ and $2\frac{1}{2}$.

Let's look at a quick example.

Below are three whole pizzas that are each cut into four pieces. A fourth pizza is there as well, but someone has taken one piece, leaving only three pieces.



You can use fractions to compare the number of pieces you have to the number of pieces that make up a *whole*. In this picture, the denominator is the total number of pieces that make up *one* whole pizza, which is 4. The total number of all pieces of pizza, which is 15, represents the numerator.

You can use the improper fraction $\frac{15}{4}$ to represent the total amount of pizza here. Think: “Each whole pizza is cut into 4 equal pieces, and there are 15 pieces total. So, the total amount of whole pizzas is $\frac{15}{4}$.”

As you looked at the image of the pizzas, however, you probably noticed right away that there were 3 full pizzas and one pizza with a piece missing. While you can use the improper fraction $\frac{15}{4}$ to represent the total amount of pizza, it makes more sense here to use a mixed number – a fraction that includes both a whole number and a fractional part. For this pizza scenario, you can use the fraction $3\frac{3}{4}$.

The number of whole pizzas you have is represented by a whole number

$3\frac{3}{4}$

← The number of pieces in the incomplete pizza is the numerator

← The total number of pieces that make up one pizza is the denominator

The mixed number $3\frac{3}{4}$ can be easier to understand than the improper fraction $\frac{15}{4}$.

However, both forms are legitimate ways to represent the number of pizzas.

Rewriting an improper fraction as a mixed number can be helpful, because it helps you see more easily about how many whole items you have.

Let’s look again at the pizzas above.

The improper fraction $\frac{15}{4}$ means there are 15 total pieces, and 4 pieces makes a whole

pizza. If you didn't have the picture, you could change $\frac{15}{4}$ into a mixed fraction by

determining:

- How many groups of 4 pieces are there in 15 pieces? Since $15 \div 4 = 3$ with a remainder, there are 3 whole pizzas.
- What is the remainder? The remainder is 3. So, there are 3 pieces of the last pizza left, out of the 4 that would make a whole pizza. So, $\frac{3}{4}$ of a pizza is left.

Now, put the number of whole pizzas with the fraction of a pizza that is left over. The mixed number is $3\frac{3}{4}$.

Writing Improper Fractions as Mixed Numbers

Step 1: Divide the denominator into the numerator.

Step 2: The quotient is the whole number part of the mixed number.

Step 3: The remainder is the numerator of the fractional part of the mixed number.

Step 4: The divisor is the denominator of the fractional part of the mixed number.

Example

Problem

Write the improper fraction $\frac{47}{7}$ as a mixed number.

$$47 \div 7 = 6, \text{ remainder } 5$$

Quotient

$$6\frac{5}{7}$$

Remainder

Divisor and original denominator

Divide the denominator into the numerator.

The quotient, 6, becomes the whole number.

The remainder, 5, becomes the numerator.

The denominator, which is also used as the divisor, remains as 7.

Answer

$$\frac{47}{7} = 6\frac{5}{7}$$

Self Check B

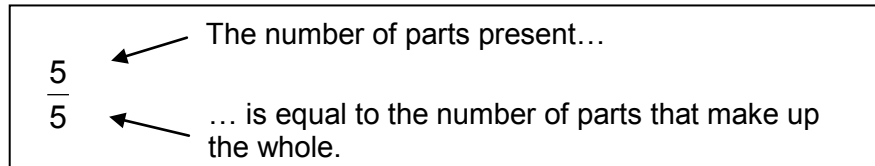
Change $\frac{12}{5}$ from an improper fraction to a mixed number.

Changing Mixed Numbers to Improper Fractions

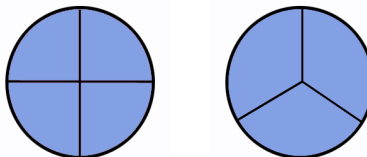
Objective 3

Mixed numbers can also be changed to improper fractions. This is sometimes helpful when doing calculations with mixed numbers, especially multiplication.

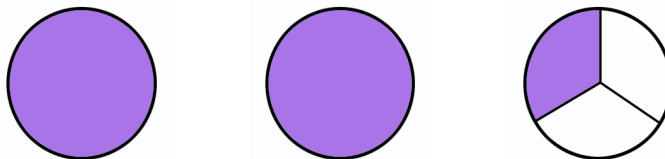
Let's start by considering the idea of one whole as an improper fraction. If you divide a cake into five equal slices, and keep all the slices, the one whole cake is equal to the 5 slices. So, 1 cake is the same as $\frac{5}{5}$ cake.



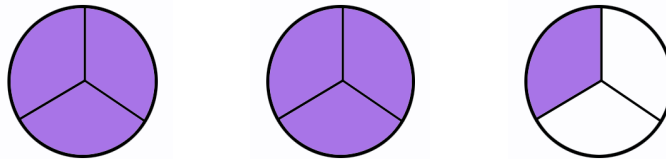
Had you cut the cake into 4 pieces or 3 pieces, as shown below, you could have used the fractions $\frac{4}{4}$ or $\frac{3}{3}$ to represent the whole cake. The fractions may change depending on the number of cuts you make to the cake, but you are still dealing with only one cake.



Let's explore how to write a simple mixed number, $2\frac{1}{3}$, as an improper fraction. The mixed number is represented below. Each full circle represents one whole.



To write an improper fraction, you need to know how many *equal sized pieces* make one whole. You also need to know how many of those pieces you have. Since you have $\frac{1}{3}$, you should divide up all of the circles into 3 pieces.



Each whole circle has 3 pieces. You can multiply the number of whole circles, 2, by 3 to find how many one-third pieces are in the two whole circles. Then you add 1 for the one-third piece in the final, incomplete circle. As you can see from the diagram, there are 7 individual one-third pieces. The improper fraction for $2\frac{1}{3}$ is $\frac{7}{3}$.

Writing Mixed Numbers as Improper Fractions

Step 1. Multiply the denominator of the fraction by the whole number.

Step 2. Add this product to the numerator of the fraction.

Step 3. The sum is the numerator of the improper fraction.

Step 4. The denominator of the improper fraction is the same as the denominator of the fractional part of the mixed number.

Example

Problem Write $4\frac{3}{4}$ as an improper fraction.

$$4\frac{3}{4}$$

$$4 \cdot 4 = 16$$

$$16 + 3 = 19$$

$$\frac{19}{4}$$

Answer: $4\frac{3}{4} = \frac{19}{4}$

Multiply the denominator of the fraction by the whole number.

Add this result to the numerator of the fraction.

This answer becomes the numerator of the improper fraction.

Notice that the denominator of the improper fraction is the same as the denominator that was in the fractional part of the mixed number

Self Check C

Change $3\frac{5}{6}$ from a mixed number to an improper fraction.

Summary

A fraction can be identified as proper or improper by comparing the numerator and the denominator. Fractions that are less than one are known as proper fractions, and the numerator (the top number) is less than the denominator (the bottom number). A fraction with a numerator that is greater than or equal to the denominator is known as an improper fraction. It represents a number greater than or equal to one. Numbers that are not whole numbers, but are greater than one, can be written as improper fractions or mixed numbers. A mixed number has a whole number part and a fraction part.

2.1.2 Self Check Solutions

Self Check A

Identify $\frac{5}{3}$ as a proper or improper fraction.

The fraction is greater than 1, and the numerator is greater than the denominator, so $\frac{5}{3}$ is an improper fraction.

Self Check B

Change $\frac{12}{5}$ from an improper fraction to a mixed number.

The improper fraction $\frac{12}{5}$ can be thought of as $12 \div 5 = 2$, with a remainder of 2. So, $2\frac{2}{5}$ is the correct answer.

Self Check C

Change $3\frac{5}{6}$ from a mixed number to an improper fraction.

$3\frac{5}{6} = (6 \cdot 3) + 5 = 23$. The denominator stays the same, so $\frac{23}{6}$ is the improper form.