Intermediate Algebra

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CHAPTER 1 Factoring Polynomials and Polynomial Equations

Chapter Outline

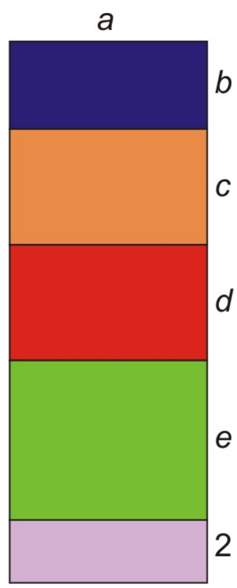
1.1 Polynomial Equations in Factored Form

Learning Objectives

- Use the zero-product property
- Find greatest common monomial factor
- Solve simple polynomial equations by factoring

Introduction

In the last few sections, we learned how to multiply polynomials. We did that by using the Distributive Property. All the terms in one polynomial must be multiplied by all terms in the other polynomial. In this section, you will start learning how to do this process in reverse. The reverse of distribution is called **factoring**.



Lets look at the areas of the rectangles again: Area = length · width. The total area of the figure on the right can be found in two ways.

Method 1 Find the areas of all the small rectangles and add them

Blue rectangle $= ab$

Orange rectangle $= ac$

Red rectangle $= ad$

Green rectangle $= ae$

Pink rectangle $= 2a$

Total area $= ab+ac+ad+ae+2a$

Method 2 Find the area of the big rectangle all at once

$$\begin{aligned}\text{Length} &= a\\ \text{Width} &= b+c+d+e+2\\ \text{Area} &= a(b+c+d+e=2)\end{aligned}$$

Since the area of the rectangle is the same no matter what method you use then the answers are the same:

$$ab+ac+ad+ae+2a = a(b+c+d+e+2)$$

Factoring means that you take the factors that are common to all the terms in a polynomial. Then, multiply them by a parenthesis containing all the terms that are left over when you divide out the common factors.

Use the Zero-Product Property

Polynomials can be written in **expanded form** or in **factored form**. Expanded form means that you have sums and differences of different terms:

$$6x^4+7x^3-26x^2+17x+30$$

Notice that the degree of the polynomials is four. It is written in standard form because the terms are written in order of decreasing power.

Factored form means that the polynomial is written as a product of different factors. The factors are also polynomials, usually of lower degree. Here is the same polynomial in factored form.

$$\underbrace{x-1}_{1^{st}\ factor}\ \underbrace{x+2}_{2^{nd}\ factor}\ \underbrace{2x-3}_{3^{rd}\ factor}\ \underbrace{3x+5}_{4^{th}\ factor}$$

Notice that each factor in this polynomial is a binomial. Writing polynomials in factored form is very useful because it helps us solve polynomial equations. Before we talk about how we can solve polynomial equations of degree 2 or higher, lets review how to solve a linear equation (degree 1).

Example 1

Solve the following equations

a) $x-4=0$

b) $3x-5=0$

Solution

Remember that to solve an equation you are trying to find the value of x:

a)

$$\begin{aligned} x-4 &= 0\\ +4 &= +4\\ \hline x &= 4 \end{aligned}$$

b)

$$\begin{aligned} 3x\ -5\ &=\ 0 \\ \underline{\quad +5\ } &\underline{=+5} \\ 3x\ &=\ 5 \\ \frac{3x}{3} &= \frac{5}{3} \\ x &= \frac{5}{3} \end{aligned}$$

Now we are ready to think about solving equations like $2x^2+5x=42$. Notice we can't isolate x in any way that you have already learned. But, we can subtract 42 on both sides to get $2x^2+5x-42=0$. Now, the left hand side of this equation can be factored!

Factoring a polynomial allows us to break up the problem into easier chunks. For example, $2x^2+5x-42=(x+6)(2x-7)$. So now we want to solve: $(x+6)(2x-7)=0$

How would we solve this? If we multiply two numbers together and their product is zero, what can we say about these numbers? The only way a product is zero is if one or both of the terms are zero. This property is called the **Zero-product Property**.

How does that help us solve the polynomial equation? Since the product equals 0, then either of the terms or factors in the product must equal zero. We set each factor equal to zero and we solve.

$$(x+6)=0 \qquad \text{OR} \qquad (2x-7)=0$$

We can now solve each part individually and we obtain:

$$\begin{array}{ccc} x+6=0 & \text{or} & 2x-7=0 \\ & & 2x=7 \\ x=-6 & \text{or} & x=-\frac{7}{2} \end{array}$$

Notice that the solution is $x=-6$**OR** $x=\frac{7}{2}$. The **OR** says that either of these values of x would make the product of the two factors equal to zero. Lets plug the solutions back into the equation and check that this is correct.

$$\begin{aligned} \text{Check } x &= -6; \\ (x+6)(2x-7) &= \\ (-6+6)(2(6)-7) &= \\ (0)(5) &= 0 \end{aligned}$$

$$\text{Check } x = \frac{7}{2}$$
$$(x+6)(2x-7) =$$
$$\left(\frac{7}{2}+6\right)\left(2\cdot\frac{7}{2}-7\right) =$$
$$\left(\frac{19}{2}\right)(7-7) =$$
$$\left(\frac{19}{2}\right)(0) = 0$$

Both solutions check out. You should notice that the product equals to zero because each solution makes one of the factors simplify to zero. Factoring a polynomial is very useful because the Zero-product Property allows us to break up the problem into simpler separate steps.

If we are not able to factor a polynomial the problem becomes harder and we must use other methods that you will learn later.

As a last note in this section, keep in mind that the Zero-product Property only works when a product equals to zero. For example, if you multiplied two numbers and the answer was nine you could not say that each of the numbers was nine. In order to use the property, you must have the factored polynomial equal to zero.

Example 2

Solve each of the polynomials

a) $(x-9)(3x+4) = 0$

b) $x(5x-4) = 0$

c) $4x(x+6)(4x-9) = 0$

Solution

Since all polynomials are in factored form, we set each factor equal to zero and solve the simpler equations separately

a) $(x-9)(3x+4) = 0$ can be split up into two linear equations

$$\begin{array}{ccc} x-9=0 & \text{or} & 3x+4=0 \\ & & 3x=-4 \\ x=9 & \text{or} & x=-\frac{4}{3} \end{array}$$

b) $x(5x-4) = 0$ can be split up into two linear equations

$$\begin{array}{ccc} & & 5x-4=0 \\ x=0 & \text{or} & 5x=4 \\ & & x=\frac{4}{5} \end{array}$$

c) $4x(x+6)(4x-9) = 0$ can be split up into three linear equations.

$$4x = 0 \qquad\qquad\qquad\qquad 4x - 9 = 0$$
$$x = \frac{0}{4} \quad \text{or} \quad x + 6 = 0 \quad \text{or} \quad 4x = 9$$
$$x = 0 \qquad\qquad x = -6 \qquad\qquad x = \frac{9}{4}$$

Find Greatest Common Monomial Factor

Once we get a polynomial in factored form, it is easier to solve the polynomial equation. But first, we need to learn how to factor. There are several factoring methods you will be learning in the next few sections. In most cases, factoring takes several steps to complete because we want to **factor completely**. That means that we factor until we cannot factor anymore.

Lets start with the simplest case, finding the greatest monomial factor. When we want to factor, we always look for common monomials first. Consider the following polynomial, written in expanded form.

$$ax + bx + cx + dx$$

A common factor can be a number, a variable or a combination of numbers and variables that appear in all terms of the polynomial. We are looking for expressions that divide out evenly from each term in the polynomial. Notice that in our example, the factor x appears in all terms. Therefore x**is a common factor**

$$ax + bx + cx + dx$$

Since x is a common factor, we factor it by writing in front of a parenthesis:

$$x(\quad)$$

Inside the parenthesis, we write what is left over when we divide x from each term.

$$x(a + b + c + d)$$

Lets look at more examples.

Example 3

Factor

a) $2x + 8$

b) $15x - 25$

c) $3a + 9b + 6$

Solution

a) We see that the factor 2 divides evenly from both terms.

$2x + 8 = 2(x) + 2(4)$

We factor the 2 by writing it in front of a parenthesis.

2( )

Inside the parenthesis, we write what is left from each term when we divide by 2.

$2(x+4)$ This is the factored form.

b) We see that the factor of 5 divides evenly from all terms.

Rewrite $15x-25=5(3x)-5(5)$

Factor 5 to get $5(3x-5)$

c) We see that the factor of 3 divides evenly from all terms.

Rewrite $3a+9b+6=3(a)+3(3b)+3(2)$

Factor 3 to get $3(a+3b+2)$

Here are examples where different powers of the common factor appear in the polynomial

Example 4

Find the greatest common factor

a) a^3-3a^2+4a

b) $12a^4-5a^3+7a^2$

Solution

a) Notice that the factor a appears in all terms of a^3-3a^2+4a but each term has a different power of a. The common factor is the lowest power that appears in the expression. In this case the factor is a.

Lets rewrite $a^3-3a^2+4a=a(a^2)+a(-3a)+a(4)$

Factor a to get $a(a^2-3a+4)$

b) The factor a appears in all the term and the lowest power is a^2.

We rewrite the expression as $12a^4-5a^3+7a^2=12a^2\cdot a^2-5a\cdot a^2+7\cdot a^2$

Factor a^2 to get $a^2(12a^2-5a+7)$

Lets look at some examples where there is more than one common factor.

Example 5:

Factor completely

a) $3ax+9a$

b) x^3y+xy

c) $5x^3y-15x^2y^2+25xy^3$

Solution

a) Notice that 3 is common to both terms.

When we factor 3 we get $3(ax+3a)$

This is not completely factored though because if you look inside the parenthesis, we notice that a is also a common factor.

When we factor a we get $3\cdot a(x+3)$

This is the answer because there are no more common factors.

A different option is to factor ***all*** common factors at once.

Since both 3 and a are common we factor the term $3a$ and get $3a(x+3)$.

b) Notice that both x and y are common factors.

Lets rewrite the expression $x^3y+xy = xy(x^2)+xy(1)$

When we factor xy we obtain $xy(x^2+1)$

c) The common factors are $5xy$.

When we factor $5xy$ we obtain $5xy(x^2-3xy+5y^2)$

Solve Simple Polynomial Equations by Factoring

Now that we know the basics of factoring, we can solve some simple polynomial equations. We already saw how we can use the Zero-product Property to solve polynomials in factored form. Here you will learn how to solve polynomials in expanded form. These are the steps for this process.

Step 1

If necessary, **re-write** the equation in standard form such that:

Polynomial expression $= 0$

Step 2

Factor the polynomial completely

Step 3

Use the zero-product rule to set **each factor equal to zero**

Step 4

Solve each equation from step 3

Step 5

Check your answers by substituting your solutions into the original equation

Example 6

Solve the following polynomial equations

a) $x^2-2x = 0$

b) $2x^2 = 5x$

c) $9x^2y-6xy = 0$

Solution:

a) $x^2-2x = 0$

Rewrite this is not necessary since the equation is in the correct form.

Factor The common factor is x, so this factors as: $x(x-2) = 0$.

Set each factor equal to zero.

$$x = 0 \qquad \text{or} \qquad x-2 = 0$$

Solve

$$x = 0 \qquad \text{or} \qquad x = 2$$

Check Substitute each solution back into the original equation.

$$\begin{array}{llll} x = 0 & \Rightarrow & (0)^2 - 2(0) = 0 & \text{works out} \\ x = 2 & \Rightarrow & (2)^2 - 2(2) = 4 - 4 = 0 & \text{works out} \end{array}$$

Answer $x = 0, x = 2$

b) $2x^2 = 5x$

Rewrite $2x^2 = 5x \Rightarrow 2x^2 - 5x = 0$.

Factor The common factor is x, so this factors as: $x(2x - 5) = 0$.

Set each factor equal to zero:.

$$x = 0 \qquad \text{or} \qquad 2x - 5 = 0$$

Solve

$$\begin{array}{ccc} \underline{x = 0} & \text{or} & 2x = 5 \\ & & x = \frac{5}{2} \end{array}$$

Check Substitute each solution back into the original equation.

$$\begin{array}{ll} x = 0 \Rightarrow 2(0)^2 = 5(0) \Rightarrow 0 = 0 & \text{works out} \\ x = \frac{5}{2} \Rightarrow 2\left(\frac{5}{2}\right)^2 = 5 \cdot \frac{5}{2} \Rightarrow 2 \cdot \frac{25}{4} = \frac{25}{2} \Rightarrow \frac{25}{2} = \frac{25}{2} & \text{works out} \end{array}$$

Answer $x = 0, x = \frac{5}{2}$

c) $9x^2y - 6xy = 0$

Rewrite Not necessary

Factor The common factor is $3xy$, so this factors as $3xy(3x - 2) = 0$.

Set each factor equal to zero.

$3 = 0$ is never true, so this part does not give a solution

$$x = 0 \qquad \text{or} \qquad y = 0 \qquad \text{or} \qquad 3x - 2 = 0$$

Solve

$$x = 0 \qquad \text{or} \qquad y = 0 \qquad \text{or} \qquad 3x = 2$$
$$amp; x = \frac{2}{3}$$

Check Substitute each solution back into the original equation.

$$x = 0 \Rightarrow 9(0)y - 6(0)y = 0 - 0 = 0 \qquad \text{works out}$$
$$y = 0 \Rightarrow 9x^2(0) - 6x = 0 - 0 = 0 \qquad \text{works out}$$
$$\frac{2}{3} \Rightarrow 9 \cdot \left(\frac{2}{3}\right)^2 y - 6 \cdot \frac{2}{3} y = 9 \cdot \frac{4}{9} y - 4y = 4y - 4y = 0 \qquad \text{works out}$$

Answer $x = 0, y = 0, x = \frac{2}{3}$

Review Questions

Factor the common factor in the following polynomials.

1. $3x^3 - 21x$
2. $5x^6 + 15x^4$
3. $4x^3 + 10x^2 - 2x$
4. $-10x^6 + 12x^5 - 4x^4$
5. $12xy + 24xy^2 + 36xy^3$
6. $5a^3 - 7a$
7. $45y^{12} + 30y^{10}$
8. $16xy^{2z} + 4x^3y$

Solve the following polynomial equations.

9. $x(x+12) = 0$
10. $(2x+1)(2x-1) = 0$
11. $(x-5)(2x+7)(3x-4) = 0$
12. $2x(x+9)(7x-20) = 0$
13. $18y - 3y^2 = 0$
14. $9x^2 = 27x$
15. $4a^2 + a = 0$
16. $b^2 - \frac{5}{3b} = 0$

Review Answers

1. $3x(x^2 - 7)$
2. $5x^4(x^2 + 3)$
3. $2x(2x^2 + 5x - 1)$
4. $2x^4(-5x^2 + 6x - 2)$
5. $12xy(1 + 2y + 3y^2)$

6. $a(5a^2-7)$
7. $15y^{10}(3y^2+2)$
8. $4xy(4yz+x^2)$
9. $x=0, x=-12$
10. $x=-\frac{1}{2}, x=\frac{1}{2}$
11. $x=5, x=-\frac{7}{2}, x=\frac{4}{3}$
12. $x=0, x=-9, x=\frac{20}{7}$
13. $y=0, y=6$
14. $x=0, x=3$
15. $a=0, a=-\frac{1}{4}$
16. $b=0, b=\frac{5}{3}$

1.2 Factoring Quadratic Expressions and Solving Quadratic Equations by Factoring

Learning Objectives

- Write quadratic equations in standard form.
- Factor quadratic expressions for different coefficient values.
- Factor when $a = -1$.
- Solve polynomial equations by factoring

Write Quadratic Expressions in Standard Form

Quadratic polynomials are polynomials of 2^{nd} degree. The standard form of a quadratic polynomial is written as

$$ax^2 + bx + c$$

Here a, b, and c stand for constant numbers. Factoring these polynomials depends on the values of these constants. In this section, we will learn how to factor quadratic polynomials for different values of a, b, and c. In the last section, we factored common monomials, so you already know how to factor quadratic polynomials where $c = 0$.

For example for the quadratic $ax^2 + bx$, the common factor is x and this expression is factored as $x(ax + b)$. When all the coefficients are not zero these expressions are also called **Quadratic Trinomials**, since they are polynomials with three terms.

Factor when a = 1, b is Positive, and c is Positive

Lets first consider the case where $a = 1, b$ is positive and c is positive. The quadratic trinomials will take the following form.

$$x^2 + bx + c$$

You know from multiplying binomials that when you multiply two factors $(x + m)(x + n)$ you obtain a quadratic polynomial. Lets multiply this and see what happens. We use The Distributive Property.

$$(x + m)(x + n) = x^2 + nx + mx + mn$$

To simplify this polynomial we would combine the like terms in the middle by adding them.

$$(x + m)(x + n) = x^2 + (n + m)x + mn$$

To factor we need to do this process in reverse.

We see that	$x^2+(n+m)x+mn$
Is the same form as	x^2+bx+c

This means that we need to find two numbers m and n where

$$n+m=b \qquad \text{and} \qquad mn=c$$

To factor x^2+bx+c, the answer is the product of two parentheses.

$(x+m)(x+n)$

so that $n+m=b$ and $mn=c$

Lets try some specific examples.

Example 1

Factor x^2+5x+6

Solution We are looking for an answer that is a product of two binomials in parentheses.

$$(x+\_\_\_)(x+\_\_\_)$$

To fill in the blanks, we want two numbers m and n that multiply to 6 and add to 5. A good strategy is to list the possible ways we can multiply two numbers to give us 6 and then see which of these pairs of numbers add to 5. The number six can be written as the product of.

$6=1\cdot 6$	and	$1+6=7$		
$6=2\cdot 3$	and	$2+3=5$	$\leftarrow$	This is the correct choice.

So the answer is $(x+2)(x+3)$.

We can check to see if this is correct by multiplying $(x+2)(x+3)$.

$$\begin{array}{rrr} & x & +\ 2 \\ \underline{} & \underline{x} & \underline{+\ 3} \\ & 3x & +\ 6 \\ \underline{x^2} & \underline{+\ 2x} & \\ x^2 & +\ 5x & +\ 9 \end{array}$$

The answer checks out.

Example 2

Factor $x^2+7x+12$

Solution

We are looking for an answer that is a product of two parentheses $(x+\_\_\_)(x+\_\_\_)$.

The number 12 can be written as the product of the following numbers.

$$\begin{array}{lllll} 12 = 1 \cdot 12 & \text{and} & 1 + 12 = 13 & & \\ 12 = 2 \cdot 6 & \text{and} & 2 + 6 = 8 & & \\ 12 = 3 \cdot 4 & \text{and} & 3 + 4 = 7 & \leftarrow & \text{This is the correct choice.} \end{array}$$

The answer is $(x+3)(x+4)$.

Example 3

Factor $x^2 + 8x + 12$.

Solution

We are looking for an answer that is a product of the two parentheses $(x+\_\_\_)(x+\_\_\_)$.

The number 12 can be written as the product of the following numbers.

$$\begin{array}{lllll} 12 = 1 \cdot 12 & \text{and} & 1 + 12 = 13 & & \\ 12 = 2 \cdot 6 & \text{and} & 2 + 6 = 8 & \leftarrow & \text{This is the correct choice.} \\ 12 = 3 \cdot 4 & \text{and} & 3 + 4 = 7 & & \end{array}$$

The answer is $(x+2)(x+6)$.

Example 4

Factor $x^2 + 12x + 36$.

Solution

We are looking for an answer that is a product of the two parentheses $(x+\_\_\_)(x+\_\_\_)$.

The number 36 can be written as the product of the following numbers.

$$\begin{array}{lllll} 36 = 1 \cdot 36 & \text{and} & 1 + 36 = 37 & & \\ 36 = 2 \cdot 18 & \text{and} & 2 + 18 = 20 & & \\ 36 = 3 \cdot 12 & \text{and} & 3 + 12 = 15 & & \\ 36 = 4 \cdot 9 & \text{and} & 4 + 9 = 13 & & \\ 36 = 6 \cdot 6 & \text{and} & 6 + 6 = 12 & \leftarrow & \text{This is the correct choice} \end{array}$$

The answer is $(x+6)(x+6)$.

Factor when a = 1, b is Negative and c is Positive

Now lets see how this method works if the middle coefficient (b) is negative.

Example 5

Factor $x^2 - 6x + 8$.

Solution

We are looking for an answer that is a product of the two parentheses $(x+\_\_\_)(x+\_\_\_)$.

The number 8 can be written as the product of the following numbers.

$8 = 1 \cdot 8$ and $1+8 = 9$ Notice that these are two different choices.

But also,

$$\begin{array}{llll} 8 = (-1)\cdot(-8) & \text{and} & -1+(-8) = -9 & \text{Notice that these are two different choices.} \\ 8 = 2\cdot 4 & \text{and} & 2+4 = 6 & \end{array}$$

But also,

$$8 = (-2)\cdot(-4) \qquad \text{and} \qquad -2+(-4) = -6 \qquad \leftarrow \qquad \text{This is the correct choice.}$$

The answer is $(x-2)(x-4)$

We can check to see if this is correct by multiplying $(x-2)(x-4)$.

$$\begin{array}{rrr} & x & -\ 2 \\ \underline{\quad} & \underline{x} & \underline{-\ 4} \\ & -\ 4x & +\ 8 \\ \underline{x^2} & \underline{-\ 2x} & \\ x^2 & -\ 6x & +\ 8 \end{array}$$

The answer checks out.

Example 6

Factor $x - 17x + 16$

Solution

We are looking for an answer that is a product of two parentheses: $(x \pm \_\_\_)(x \pm \_\_\_)$.

The number 16 can be written as the product of the following numbers:

$$\begin{array}{llll} 16 = 1\cdot 16 & \text{and} & 1+16 = 17 & \\ 16 = (-1)\cdot(-16) & \text{and} & -1+(-16) = -17 & \leftarrow \quad \text{This is the correct choice.} \\ 16 = 2\cdot 8 & \text{and} & 2+8 = 10 & \\ 16 = (-2)\cdot(-8) & \text{and} & -2+(-8) = -10 & \\ 16 = 4\cdot 4 & \text{and} & 4+4 = 8 & \\ 16 = (-4)\cdot(-4) & \text{and} & -4+(-4) = -8 & \end{array}$$

The answer is $(x-1)(x-16)$.

Factor when a = 1 and c is Negative

Now lets see how this method works if the constant term is negative.

Example 7

Factor $x^2+2x-15$

Solution

We are looking for an answer that is a product of two parentheses $(x \pm \_\_\_)(x \pm \_\_\_)$.

In this case, we must take the negative sign into account. The number -15 can be written as the product of the following numbers.

$-15=-1\cdot 15$ and $-1+15=14$ Notice that these are two different choices.

And also,

$-15=1\cdot(-15)$ and $1+(-15)=-14$ Notice that these are two different choices.

$$\begin{array}{lll} -15=-3\cdot 5 & \text{and} & -3+5=2 \quad \leftarrow \quad \text{This is the correct choice.} \\ -15=3\cdot(-5) & \text{and} & 3+(-5)=-2 \end{array}$$

The answer is $(x-3)(x+5)$.

We can check to see if this is correct by multiplying $(x-3)(x+5)$.

$$\begin{array}{rrr} & x & -\ 3 \\ & x & +\ 5 \\ \hline & 5x & -\ 15 \\ x^2 & -\ 3x & \\ \hline x^2 & +\ 2x & -\ 15 \end{array}$$

The answer checks out.

Example 8

Factor $x^2-10x-24$

Solution

We are looking for an answer that is a product of two parentheses $(x \pm \_\_\_)(x \pm \_\_\_)$.

The number -24 can be written as the product of the following numbers.

$$\begin{array}{lll} -24=-1\cdot 24 & \text{and} & -1+24=23 \\ -24=1\cdot(-24) & \text{and} & 1+(-24)=-23 \\ -24=-2\cdot 12 & \text{and} & -2+12=10 \\ -24=2\cdot(-12) & \text{and} & 2+(-12)=-10 \quad \leftarrow \quad \text{This is the correct choice.} \\ -24=-3\cdot 8 & \text{and} & -3+8=5 \\ -24=3\cdot(-8) & \text{and} & 3+(-8)=-5 \\ -24=-4\cdot 6 & \text{and} & -4+6=2 \\ -24=4\cdot(-6) & \text{and} & 4+(-6)=-2 \end{array}$$

The answer is $(x-12)(x+2)$.

Example 9

Factor $x^2+34x-35$

Solution

We are looking for an answer that is a product of two parentheses $(x\pm \_\_\_)(x\pm \_\_\_)$

The number -35 can be written as the product of the following numbers:

$$\begin{array}{llll} -35=-1\cdot 35 & \text{and} & -1+35=34 & \leftarrow \quad \text{This is the correct choice.} \\ -35=1\cdot(-35) & \text{and} & 1+(-35)=-34 & \\ -35=-5\cdot 7 & \text{and} & -5+7=2 & \\ -35=5\cdot(-7) & \text{and} & 5+(-7)=-2 & \end{array}$$

The answer is $(x-1)(x+35)$.

Factor when a = - 1

When $a=-1$, the best strategy is to factor the common factor of -1 from all the terms in the quadratic polynomial. Then, you can apply the methods you have learned so far in this section to find the missing factors.

Example 10

Factor $-x^2+x+6$.

Solution

First factor the common factor of -1 from each term in the trinomial. Factoring -1 changes the signs of each term in the expression.

$$-x^2+x+6=-(x^2-x-6)$$

We are looking for an answer that is a product of two parentheses $(x\pm \_\_\_)(x\pm \_\_\_)$

Now our job is to factor x^2-x-6.

The number -6 can be written as the product of the following numbers.

$$\begin{array}{llll} -6=-1\cdot 6 & \text{and} & -1+6=5 & \\ -6=1\cdot(-6) & \text{and} & 1+(-6)=-5 & \\ -6=-2\cdot 3 & \text{and} & -2+3=1 & \\ -6=2\cdot(-3) & \text{and} & 2+(-3)=-1 & \leftarrow \quad \text{This is the correct choice.} \end{array}$$

The answer is $-(x-3)(x+2)$.

Solve Quadratic Equations by Factoring

Now that we know the basics of factoring, we can solve some simple polynomial equations. We already saw how we can use the Zero-product Property to solve polynomials in factored form. Here you will learn how to solve polynomials in expanded form. These are the steps for this process.

Step 1

If necessary, **re-write** the equation in standard form such that:

Polynomial expression $= 0$

Step 2

Factor the quadratic completely

Step 3

Use the zero-product rule to set **each factor equal to zero**

Step 4

Solve each equation from step 3

Step 5

Check your answers by substituting your solutions into the original equation

Example 11

Solve the following polynomial equations

a) $x^2 - 2x - 15 = 0$

b) $x^2 = 5x + 6$

c) $-x^2 = 8 - 6x$

Solution:

a) $x^2 - 2x - 15 = 0$

Rewrite this is not necessary since the equation is in the correct form.

Factor $(x-5)(x+3) = 0$.

Set each factor equal to zero.

$$x - 5 = 0 \qquad \text{or} \qquad x + 3 = 0$$

Solve

$$x = 5 \qquad \text{or} \qquad x = -3$$

Check Substitute each solution back into the original equation.

$$\begin{array}{llll} x = 5 & \Rightarrow & (5)^2 - 2(5) - 15 = 0 & \text{works out} \\ x = -3 & \Rightarrow & (-3)^2 - 2(-3) - 15 = 0 & \text{works out} \end{array}$$

Answer $x = 5, x = -3$

b) $x^2 = 5x + 6$

Rewrite $x^2 - 5x - 6 = 0$.

Factor $(x-6)(x+1) = 0$.

Set each factor equal to zero:.

$$x-6=0 \quad \text{or} \quad x+1=0$$

Solve

$$x=6 \qquad \text{or} \qquad x=-1$$

Check Substitute each solution back into the original equation.

$$x=6 \Rightarrow (6)^2-5(6)-6=0 \qquad \text{works out}$$
$$x=-1 \Rightarrow (-1)^2-5(-1)-6=0 \qquad \text{works out}$$

Answer $x=6, x=-1$

c) $-x^2=8-6x$

Rewrite $-x^2+6x-8=0$

Factor

$$-x^2+6x-8=0 \Rightarrow -1(x^2-6x+8)=0 \Rightarrow -1(x-4)(x-2)=0$$

.

Set each factor equal to zero:.

$$x-4=0 \quad \text{or} \quad x-2=0$$

Solve

$$x=4 \qquad \text{or} \qquad x=2$$

Check Substitute each solution back into the original equation.

$$x=4 \Rightarrow -(4)^2+6(4)-8=0 \qquad \text{works out}$$
$$x=2 \Rightarrow -(2)^2+6(2)-8=0 \qquad \text{works out}$$

Answer $x=4, x=2$

To Summarize,

A quadratic of the form $x+bx+c$ factors as a product of two parenthesis $(x+m)(x+n)$.

- If b and c are positive then both m and $n$are positive
  - Example $x^2+8x+12$ factors as $(x+6)(x+2)$.
- If b is negative and c is positive then both m and $n$are negative.

  - Example x^2-6x+8 factors as $(x-2)(x-4)$.
- If c is negative then either m is positive and $n$is negative or vice-versa
  - Example $x^2+2x-15$ factors as $(x+5)(x-3)$.
  - Example $x^2+34x-35$ factors as $(x+35)(x-1)$.
- If $a=-1$, factor a common factor of -1 from each term in the trinomial and then factor as usual. The answer will have the form $-(x+m)(x+n)$.
  - Example $-x^2+x+6$ factors as $-(x-3)(x+2)$.

Review Questions

Factor the following quadratic polynomials.

1. $x^2+10x+9$
2. $x^2+15x+50$
3. $x^2+10x+21$
4. $x^2+16x+48$
5. $x^2-11x+24$
6. $x^2-13x+42$
7. $x^2-14x+33$
8. $x^2-9x+20$
9. $x^2+5x-14$
10. $x^2+6x-27$
11. $x^2+7x-78$
12. $x^2+4x-32$
13. $x^2-12x-45$
14. $x^2-5x-50$
15. $x^2-3x-40$
16. x^2-x-56
17. $-x^2-2x-1$
18. $-x^2-5x+24$
19. $-x^2+18x-72$
20. $-x^2+25x-150$
21. $x^2+21x+108$
22. $-x^2+11x-30$
23. $x^2+12x-64$
24. $x^2-17x-60$

Solve.

25. $x^2+12x-28=0$
26. $x^2-8x+12=0$
27. $x^2-9x-36=0$
28. $-x^2=x-20$

Review Answers

1. $(x+1)(x+9)$
2. $(x+5)(x+10)$

3. $(x+7)(x+3)$
4. $(x+12)(x+4)$
5. $(x-3)(x-8)$
6. $(x-7)(x-6)$
7. $(x-11)(x-3)$
8. $(x-5)(x-4)$
9. $(x-2)(x+7)$
10. $(x-3)(x+9)$
11. $(x-6)(x+13)$
12. $(x-4)(x+8)$
13. $(x-15)(x+3)$
14. $(x-10)(x+5)$
15. $(x-8)(x+5)$
16. $(x-8)(x+7)$
17. $-(x+1)(x+1)$
18. $-(x-3)(x+8)$
19. $-(x-6)(x-12)$
20. $-(x-15)(x-10)$
21. $(x+9)(x+12)$
22. $-(x-5)(x-6)$
23. $(x-4)(x+16)$
24. $(x-20)(x+3)$
25. $x=-14, x=2$
26. $x=6, x=2$
27. $x=12, x=-3$
28. $x=4, x=-5$

1.3 Factoring Special Products and Solving Quadratic Equations by Factoring

Learning Objectives

- Factor the difference of two squares.
- Factor perfect square trinomials.
- Factor the sum and difference of cubes.
- Solve quadratic polynomial equation by factoring.

Introduction

When you learned how to multiply binomials we talked about two special products.

The Sum and Difference Formula	$a^2-b^2=(a+b)(a-b)$
The Square of a Binomial Formula	$a^2+2ab+b^2=(a+b)^2$
	$a^2-2ab+b^2=(a-b)^2$
The Sum or Difference of Cubes Formula	$a^3+b^3=(a+b)(a^2-ab+b^2)$
	$a^3-b^3=(a-b)(a^2+ab+b^2)$

In this section we will learn how to recognize and factor these special products.

Factor the Difference of Two Squares

We use the sum and difference formula to factor a difference of two squares. A difference of two squares can be a quadratic polynomial in this form.

$$a^2-b^2$$

Both terms in the polynomial are perfect squares. In a case like this, the polynomial factors into the sum and difference of the square root of each term.

$$a^2-b^2=(a+b)(a-b)$$

In these problems, the key is figuring out what the a and b terms are. Lets do some examples of this type.

Example 1

Factor the difference of squares.

a) x^2-9

b) x^2-100

c) x^2-1

Solution

a) Rewrite as $x^2 - 9$ as $x^2 - 3^2$. Now it is obvious that it is a difference of squares.

$$\begin{array}{rr} \text{The difference of squares formula is} & a^2 - b^2 = (a+b)(a-b) \\ \text{Lets see how our problem matches with the formula} & x^2 - 3^2 = (x+3)(x-3) \end{array}$$

The answer is $x^2 - 9 = (x+3)(x-3)$.

We can check to see if this is correct by multiplying $(x+3)(x-3)$.

$$\begin{array}{rrr} & x & +\ 3 \\ & x & -\ 3 \\ \hline & -3x & -\ 9 \\ x^2 & +3x & \\ \hline x^2 & +0x & -\ 9 \end{array}$$

The answer checks out.

We could factor this polynomial without recognizing that it is a difference of squares. With the methods we learned in the last section we know that a quadratic polynomial factors into the product of two binomials.

$$(x \pm \_\_\_)(x \pm \_\_\_)$$

We need to find two numbers that multiply to -9 and add to 0, since the middle term is missing.

We can write -9 as the following products

$$\begin{array}{llll} -9 = -1 \cdot 9 & \text{and} & -1+9 = 8 & \\ -9 = 1 \cdot (-9) & \text{and} & 1+(-9) = -8 & \\ -9 = 3 \cdot (-3) & \text{and} & 3+(-3) = 0 & \leftarrow \quad \text{This is the correct choice} \end{array}$$

We can factor $x^2 - 9$ as $(x+3)(x-3)$, which is the same answer as before.

You can always factor using methods for factoring trinomials, but it is faster if you can recognize special products such as the difference of squares.

b) Rewrite $x^2 - 100$ as $x^2 - 10^2$. This factors as $(x+10)(x-10)$.

c) Rewrite $x^2 - 1$ as $x^2 - 1^2$. This factors as $(x+1)(x-1)$.

Example 2

Factor the difference of squares.

a) $16x^2 - 25$

b) $4x^2 - 81$

c) $49x^2 - 64$

Solution

a) Rewrite $16x^2-25$ as $(4x)^2-5^2$. This factors as $(4x+5)(4x-5)$.

b) Rewrite $4x^2-81$ as $(2x)^2-9^2$. This factors as $(2x+9)(2x-9)$.

c) Rewrite $49x^2-64$ as $(7x)^2-8^2$. This factors as $(7x+8)(7x-8)$.

Example 3

Factor the difference of squares:

a) x^2-y^2

b) $9x^2-4y^2$

c) x^2y^2-1

Solution

a) x^2-y^2 factors as $(x+y)(x-y)$.

b) Rewrite $9x^2-4y^2$ as $(3x)^2-(2y)^2$. This factors as $(3x+2y)(3x-2y)$.

c) Rewrite as x^2y^2-1 as $(xy)^2-1^2$. This factors as $(xy+1)(xy-1)$.

Example 4

Factor the difference of squares.

a) x^4-25

b) $16x^4-y^2$

c) $x^2y^8-64z^2$

Solution

a) Rewrite x^4-25 as $(x^2)^2-5^2$. This factors as $(x^2+5)(x^2-5)$.

b) Rewrite $16x^4-y^2$ as $(4x^2)^2-y^2$. This factors as $(4x^2+y)(4x^2-y)$

c) Rewrite $x^2y^8-64z^2$ as $(xy^2)^2-(8z)$. This factors as $(xy^2+8z)(xy^2-8z)$.

Factor Perfect Square Trinomials

We use the **Square of a Binomial Formula** to factor perfect square trinomials. A perfect square trinomial has the following form.

$$a^2+2ab+b^2 \qquad \text{or} \qquad a^2-2ab+b^2$$

In these special kinds of trinomials, the first and last terms are perfect squares and the middle term is twice the product of the square roots of the first and last terms. In a case like this, the polynomial factors into perfect squares.

$$a^2+2ab+b^2=(a+b)^2$$
$$a^2-2ab+b^2=(a-b)^2$$

In these problems, the key is figuring out what the a and b terms are. Lets do some examples of this type.

Example 5

Factor the following perfect square trinomials.

a) $x^2+8x+16$

b) $x^2 - 4x + 4$

c) $x^2 + 14x + 49$

Solution

a) $x^2 + 8x + 16$

The first step is to recognize that this expression is actually perfect square trinomials.

1. Check that the first term and the last term are perfect squares. They are indeed because we can re-write:

$$x^2 + 8x + 16 \qquad \text{as} \qquad x^2 + 8x + 4^2.$$

2. Check that the middle term is twice the product of the square roots of the first and the last terms. This is true also since we can rewrite them.

$$x^2 + 8x + 16 \qquad \text{as} \qquad x^2 + 2 \cdot 4 \cdot x + 4^2$$

This means we can factor $x^2 + 8x + 16$ as $(x+4)^2$.

We can check to see if this is correct by multiplying $(x+4)(x+4)$.

$$\begin{array}{r} x + 4 \\ \underline{x - 4} \\ 4x + 16 \\ \underline{x^2 + 4x \quad\;\;} \\ x^2 + 8x + 16 \end{array}$$

The answer checks out.

We could factor this trinomial without recognizing it as a perfect square. With the methods we learned in the last section we know that a trinomial factors as a product of the two binomials in parentheses.

$$(x \pm \_\_\_)(x \pm \_\_\_)$$

We need to find two numbers that multiply to 16 and add to 8. We can write 16 as the following products.

$$\begin{array}{llll} 16 = 1 \cdot 16 & \text{and} & 1 + 16 = 17 & \\ 16 = 2 \cdot 8 & \text{and} & 2 + 8 = 10 & \\ 16 = 4 \cdot 4 & \text{and} & 4 + 4 = 8 & \leftarrow \quad \text{This is the correct choice.} \end{array}$$

We can factor $x^2 + 8x + 16$ as $(x+4)(x+4)$ which is the same as $(x+4)^2$.

You can always factor by the methods you have learned for factoring trinomials but it is faster if you can recognize special products.

b) Rewrite $x^2 - 4x + 4$ as $x^2 + 2 \cdot (-2) \cdot x + (-2)^2$.

We notice that this is a perfect square trinomial and we can factor it as: $(x-2)^2$.

c) Rewrite $x^2+14x+49$ as $x^2+2\cdot 7\cdot x+7^2$.

We notice that this is a perfect square trinomial as we can factor it as: $(x+7)^2$.

Example 6

Factor the following perfect square trinomials.

a) $4x^2+20x+25$

b) $9x^2-24x+16$

c) $x+2xy+y^2$

Solution

a) Rewrite $4x^2+20x+25$ as $(2x)^2+2.5\cdot(2x)+5^2$

We notice that this is a perfect square trinomial and we can factor it as $(2x+5)^2$.

b) Rewrite $9x^2-24x+16$ as $(3x)^2+2\cdot(-4)\cdot(3x)+(-4)^2$.

We notice that this is a perfect square trinomial as we can factor it as $(3x-4)^2$.

We can check to see if this is correct by multiplying $(3x-4)^2=(3x-4)(3x-4)$.

$$\begin{array}{rrrrr} & & 3x & - & 4 \\ \hline & & 3x & - & 4 \\ & - & 12x & + & 16 \\ 9x^2 & - & 12x & & \\ \hline 9x^2 & - & 24x & + & 16 \end{array}$$

The answer checks out.

c) $x+2xy+y^2$

We notice that this is a perfect square trinomial as we can factor it as $(x+y)^2$.

Factor a Sum or Difference of Cubes

We use the **Sum of Difference of Cubes Formula** to factor a sum of difference of cubes. A sum of difference of cubes has one of the following forms.

$$a^3+b^2 \qquad \text{or} \qquad a^3-b^3$$

In these special kinds of trinomials, the first and last terms are perfect squares and the middle term is twice the product of the square roots of the first and last terms. In a case like this, the polynomial factors into perfect squares.

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

In these problems, the key is figuring out what the a and b terms are. Lets do some examples of this type.

Example 7

Factor the difference of squares:

a) $x^3 - 8$

b) $27x^3 + 64y^3$

Solution

a) $x^3 - 8 = x^3 - (2)^3$ factors as $(x-2)(x^2 + 2x + 4)$.

b) Rewrite $27x^3 + 64y^3$ as $(3x)^3 - (4y)^3$. This factors as $(3x + 4y)(9x^2 - 12xy + 16y^2)$.

Solve Quadratic Polynomial Equations by Factoring

With the methods we learned in the last two sections, we can factor many kinds of quadratic polynomials. This is very helpful when we want to solve polynomial equations such as

$$ax^2 + bx + c = 0$$

Remember that to solve polynomials in expanded form we use the following steps:

Step 1

If necessary, **rewrite** the equation in standard form so that

Polynomial expression = 0.

Step 2

Factor the polynomial completely.

Step 3

Use the Zero-Product rule to **set each factor equal to zero**.

Step 4

Solve each equation from Step 3.

Step 5

Check your answers by substituting your solutions into the original equation.

We will do a few examples that show how to solve quadratic polynomials using the factoring methods we just learned.

Example 8

Solve the following polynomial equations.

a) $x^2 + 7x + 6 = 0$

b) $x^2 - 8x = -12$

c) $x^2 = 2x + 15$

Solution

a) $x^2 + 7x + 6 = 0$

Rewrite This is not necessary since the equation is in the correct form already.

Factor We can write 6 as a product of the following numbers.

$6 = 1 \cdot 6$	and	$1+6=7$	$\leftarrow$	This is the correct choice.
$6 = 2 \cdot 3$	and	$2+3=5$		

$x^2+7x+6=0$ factors as $(x+1)(x+6)=0$

Set each factor equal to zero

$$x+1=0 \qquad \text{or} \qquad x+6=0$$

Solve

$$x=-1 \qquad \text{or} \qquad x=-6$$

Check Substitute each solution back into the original equation.

$x=-1$	$(-1)^2+7(-1)+6=1-7+6=0$	Checks out.
$x=-6$	$(-6)^2+7(-6)+6=36-42+6=0$	Checks out.

b) $x^2-8x=-12$

Rewrite $x^2-8x=-12$ is rewritten as $x^2-8x+12=0$.

Factor We can write 12 as a product of the following numbers.

$12 = 1 \cdot 12$	and	$1+12=13$		
$12 = -1 \cdot (-12)$	and	$-1+(-12)=-13$		
$12 = 2 \cdot 6$	and	$2+6=8$		
$12 = -2 \cdot (-6)$	and	$-2+(-6)=-8$	$\leftarrow$	This is the correct choice.
$12 = 3 \cdot 4$	and	$3+4=7$		
$12 = -3 \cdot (-4)$	and	$-3+(-4)=-7$		

$x^2-8x+12=0$ factors as $(x-2)(x-6)=0$

Set each factor equal to zero.

$$x-2=0 \qquad \text{or} \qquad x-6=0$$

Solve.

$$x=2 \qquad \text{or} \qquad x=6$$

Check Substitute each solution back into the original equation.

$$x = 2 \qquad (2)^2 - 8(2) = 4 - 16 = -12 \qquad \text{Checks out.}$$
$$x = 2 \qquad (6)^2 - 8(6) = 36 - 48 = -12 \qquad \text{Checks out.}$$

c) $x^2 = 2x + 15$

Rewrite $x^2 = 2x + 15$ is re-written as $x^2 - 2x - 15 = 0$.

Factor We can write -15 as a product of the following numbers.

$$-15 = 1 \cdot (-15) \qquad \text{and} \qquad 1 + (-15) = -14$$
$$-15 = -1 \cdot (15) \qquad \text{and} \qquad -1 + (15) = 14$$
$$-15 = -3 \cdot 5 \qquad \text{and} \qquad -3 + 5 = 2$$
$$-15 = 3 \cdot (-5) \qquad \text{and} \qquad 3 + (-5) = -2 \qquad \leftarrow \qquad \text{This is the correct choice.}$$

$x^2 - 2x - 15 = 0$ factors as $(x + 3)(x - 5) = 0$.

Set each factor equal to zero

$$x + 3 = 0 \qquad \text{or} \qquad x - 5 = 0$$

Solve

$$x = -3 \qquad \text{or} \qquad x = 5$$

Check Substitute each solution back into the original equation.

$$x = -3 \qquad (-3)^2 = 2(-3) + 15 \Rightarrow 9 = 0 \qquad \text{Checks out.}$$
$$x = 5 \qquad (5)^2 = 2(5) + 15 \Rightarrow 25 = 25 \qquad \text{Checks out.}$$

Example 8

Solve the following polynomial equations.

a) $x^2 - 12x + 36 = 0$

b) $x^2 - 81 = 0$

c) $x^2 + 20x + 100 = 0$

Solution

a) $x^2 - 12x + 36 = 0$

Rewrite This is not necessary since the equation is in the correct form already.

Factor: Re-write $x^2 - 12x + 36$ as $x^2 - 2 \cdot (-6)x + (-6)^2$.

We recognize this as a difference of squares. This factors as $(x - 6)^2 = 0$ or $(x - 6)(x - 6) = 0$.

Set each factor equal to zero

$$x-6=0 \qquad \text{or} \qquad x-6=0$$

Solve

$$x=6 \qquad \text{or} \qquad x=6$$

Notice that for a perfect square the two solutions are the same. This is called a **double root**.

Check Substitute each solution back into the original equation.

$$x=6 \qquad 6^2-12(6)+36=36-72+36+0 \qquad \text{Checks out.}$$

b) $x^2-81=0$

Rewrite This is not necessary since the equation is in the correct form already

Factor Rewrite $x^2-81=0$ as $x^2-9^2=0$.

We recognize this as a difference of squares. This factors as $(x-9)(x+9)=0$.

Set each factor equal to zero.

$$ath=x-9=0 \qquad \text{or} \qquad x+9=0$$

Solve:

$$ath=x=9 \qquad \text{or} \qquad x=-9$$

Check: Substitute each solution back into the original equation.

$$ath=x=9 \qquad 9^2-81=81-81=0 \qquad \text{Checks out.}$$
$$x=-9 \qquad (-9)^2-81=81-81=0 \qquad \text{Checks out.}$$

c) $x^2+20x+100=0$

Rewrite This is not necessary since the equation is in the correct form already.

Factor Rewrite $x^2+20x+100=0$ as $x^2+2\cdot 10\cdot x+10^2$

We recognize this as a perfect square. This factors as: $(x+10)^2=0$ or $(x+10)(x+10)=0$.

Set each factor equal to zero.

$$ath=x+10=0 \qquad \text{or} \qquad x+10=0$$

Solve.

$$ath = x = -10 \qquad \text{or} \qquad x = -10 \qquad \text{This is a double root.}$$

Check Substitute each solution back into the original equation.

$$ath = x = 10 \qquad (-10)^2 + 20(-10) + 100 = 100 - 200 + 100 = 0 \qquad \text{Checks out.}$$

Review Questions

Factor the following perfect square trinomials.

1. $x^2 + 8x + 16$
2. $x^2 - 18x + 81$
3. $-x^2 + 24x - 144$
4. $x^2 + 14x + 49$
5. $4x^2 - 4x + 1$
6. $25x^2 + 60x + 36$
7. $4x^2 - 12xy + 9y^2$
8. $x^4 + 22x^2 + 121$

Factor the following difference of squares.

9. $x^2 - 4$
10. $x^2 - 36$
11. $-x^2 + 100$
12. $x^2 - 400$
13. $9x^2 - 4$
14. $25x^2 - 49$
15. $-36x^2 + 25$
16. $16x^2 - 81y^2$

Factor the following sum or difference of cubes.

17. $x^3 + 27$
18. $8x^3 - 1$
19. $64x^3 + 125y^3$

Solve the following quadratic equation using factoring.

20. $x^2 - 11x + 30 = 0$
21. $x^2 + 4x = 21$
22. $x^2 + 49 = 14x$
23. $x^2 - 64 = 0$
24. $x^2 - 24x + 144 = 0$
25. $4x^2 - 25 = 0$

26. $x^2+26x=-169$
27. $-x^2-16x-60=0$

Review Answers

1. $(x+4)^2$
2. $(x-9)^2$
3. $-(x-12)^2$
4. $(x+7)^2$
5. $(2x-1)^2$
6. $(5x+6)^2$
7. $(2x-3y)^2$
8. $(x^2+11)^2$
9. $(x+2)(x-2)$
10. $(x+6)(x-6)$
11. $-(x+10)(x-10)$
12. $(x+20)(x-20)$
13. $(3x+2)(3x-2)$
14. $(5x+7)(5x-7)$
15. $-(6x+5)(6x-5)$
16. $(4x+9y)(4x-9y)$
17. $(x+3)(x^2-3x+9)$
18. $(2x-1)(4x^2+2x+1)$
19. $(4x+5y)(16x^2-20xy+25y^2)$
20. $x=5, x=6$
21. $x=-7, x=3$
22. $x=7$
23. $x=-8, x=8$
24. $x=12$
25. $x=\frac{5}{2}, x=-\frac{5}{2}$
26. $x=-13$
27. $x=-10, x=-6$

1.4 Factoring Polynomials Completely and Solving Polynomial Equations by Factoring

Learning Objectives

- Factor out a common binomial.
- Factor by grouping.
- Factor a quadratic trinomial where $a \neq 1$.
- Solve polynomial equations by factoring.
- Solve real world problems using polynomial equations.

Introduction

We say that a polynomial is **factored completely** when we factor as much as we can and we cant factor any more. Here are some suggestions that you should follow to make sure that you factor completely.

- Factor all common monomials first.
- Identify special products such as difference of squares or the square of a binomial. Factor according to their formulas.
- If there are no special products, factor using the methods we learned in the previous sections.
- Look at each factor and see if any of these can be factored further.

Here are some examples

Example 1

Factor the following polynomials completely.

a) $6x^2 - 30x + 24$

b) $2x^2 - 8$

c) $x^3 + 6x^2 + 9x$

Solution

a) $6x^2 - 30x + 24$

Factor the common monomial. In this case 6 can be factored from each term.

$$6(x^2 - 5x + 6)$$

There are no special products. We factor $x^2 - 5x + 6$ as a product of two binomials $(x \pm \_\_\_)(x \pm \_\_\_)$.

The two numbers that multiply to 6 and add to -5 are -2 and -3. Let’s substitute them into the two parenthesis. The 6 is outside because it is factored out.

$$6(x^2 - 5x + 6) = 6(x - 2)(x - 3)$$

If we look at each factor we see that we can’t factor anything else.

The answer is $6(x-2)(x-3)$

b) $2x^2-8$

Factor common monomials $2x^2-8=2(x^2-4)$.

We recognize x^2-4 as a difference of squares. We factor as $2(x^2-4)=2(x+2)(x-2)$.

If we look at each factor we see that we can't factor anything else.

The answer is $2(x+2)(x-2)$.

c) x^3+6x^2+9x

Factor common monomials $x^3+6x^2+9x=x(x^2+6x+9)$.

We recognize as a perfect square and factor as $x(x+3)^2$.

If we look at each factor we see that we can't factor anything else.

The answer is $x(x+3)^2$.

Example 2

Factor the following polynomials completely.

a) $-2x^4+162$

b) x^5-8x^3+16x

Solution

a) $-2x^4+162$

Factor the common monomial. In this case, factor -2 rather than 2. It is always easier to factor the negative number so that the leading term is positive.

$$-2x^4+162=-2(x^4-81)$$

We recognize expression in parenthesis as a difference of squares. We factor and get this result.

$$-2(x^2-9)(x^2+9)$$

If we look at each factor, we see that the first parenthesis is a difference of squares. We factor and get this answers.

$$-2(x+3)(x-3)(x^2+9)$$

If we look at each factor, we see that we can factor no more.

The answer is $-2(x+3)(x-3)(x^2+9)$

b) x^5-8x^3+16x

Factor out the common monomial $x^5-8x^3+14x=x(x^4-8x^2+16)$.

We recognize x^4-8x^2+16 as a perfect square and we factor it as $x(x^2-4)^2$.

We look at each term and recognize that the term in parenthesis is a difference of squares.

We factor and get: $x[(x+2)^2(x-2)]^2=x(x+2)^2(x-2)^2$.

We use square brackets [and] in this expression because x is multiplied by the expression $(x+2)^2(x-2)$. When we have nested grouping symbols we use brackets [and] to show the levels of nesting.

If we look at each factor now we see that we can't factor anything else.

The answer is: $x(x+2)^2(x-2)^2$.

Factor out a Common Binomial

The first step in the factoring process is often factoring the common monomials from a polynomial. Sometimes polynomials have common terms that are binomials. For example, consider the following expression.

$$x(3x+2)-5(3x+2)$$

You can see that the term $(3x+2)$ appears in both term of the polynomial. This common term can be factored by writing it in front of a parenthesis. Inside the parenthesis, we write all the terms that are left over when we divide them by the common factor.

$$(3x+2)(x-5)$$

This expression is now completely factored.

Lets look at some more examples.

Example 3

Factor the common binomials.

a) $3x(x-1)+4(x-1)$

b) $x(4x+5)+(4x+5)$

Solution

a) $3x(x-1)+4(x-1)$ has a common binomial of $(x-1)$.

When we factor the common binomial, we get $(x-1)(3x+4)$.

b) $x(4x+5)+(4x+5)$ has a common binomial of $(4x+5)$.

When we factor the common binomial, we get $(4x+5)(x+1)$.

Factor by Grouping

It may be possible to factor a polynomial containing four or more terms by factoring common monomials from groups of terms. This method is called **factor by grouping**.

The next example illustrates how this process works.

Example 4

Factor $2x+2y+ax+ay$.

Solution

There isn't a common factor for all four terms in this example. However, there is a factor of 2 that is common to the first two terms and there is a factor of a that is common to the last two terms. Factor 2 from the first two terms and factor a from the last two terms.

$$2x+2y+ax+ay = 2(x+y)+a(x+y)$$

Now we notice that the binomial $(x+y)$ is common to both terms. We factor the common binomial and get.

$$(x+y)(2+a)$$

Our polynomial is now factored completely.

Example 5

Factor $3x^2+6x+4x+8$.

Solution

We factor $3x$ from the first two terms and factor 4 from the last two terms.

$$3x(x+2)+4(x+2)$$

Now factor $(x+2)$ from both terms.

$$(x+2)(3x+4).$$

Now the polynomial is factored completely.

Factor Quadratic Trinomials Where a 1

Factoring by grouping is a very useful method for factoring quadratic trinomials where $a \neq 1$. A quadratic polynomial such as this one.

$$ax^2+bx+c$$

This does not factor as $(x \pm m)(x \pm n)$, so it is not as simple as looking for two numbers that multiply to give c and add to give b. In this case, we must take into account the coefficient that appears in the first term.

To factor a quadratic polynomial where $a \neq 1$, we follow the following steps.

1. We find the product ac.
2. We look for two numbers that multiply to give ac and add to give b.
3. We rewrite the middle term using the two numbers we just found.
4. We factor the expression by grouping.

Lets apply this method to the following examples.

Example 6

Factor the following quadratic trinomials by grouping.

a) $3x^2+8x+4$

b) $6x^2-11x+4$

c) $5x^2-6x+1$

Solution

Lets follow the steps outlined above.

a) $3x^2+8x+4$

Step 1 $ac = 3 \cdot 4 = 12$

Step 2 The number 12 can be written as a product of two numbers in any of these ways:

$12 = 1 \cdot 12$	and	$1+12 = 13$	
$12 = 2 \cdot 6$	and	$2+6 = 8$	This is the correct choice.
$12 = 3 \cdot 4$	and	$3+4 = 7$	

Step 3 Re-write the middle term as: $8x = 2x+6x$, so the problem becomes the following.

$$3x^2+8x+4 = 3x^2+2x+6x+4$$

Step 4: Factor an x from the first two terms and 2 from the last two terms.

$$x(3x+2)+2(3x+2)$$

Now factor the common binomial $(3x+2)$.

$$(3x+2)(x+2)$$

Our answer is $(3x+2)(x+2)$.

To check if this is correct we multiply $(3x+2)(x+2)$.

$$\begin{array}{r} 3x+\ 2 \\ \underline{\qquad\quad x\ +\ 2} \\ 6x+\ 4 \\ \underline{3x^2\ +\ \ 2x\qquad} \\ 3x^2\ +\ 8x\ +\ 4 \end{array}$$

The answer checks out.

b) $6x^2-11x+4$

Step 1 $ac = 6 \cdot 4 = 24$

Step 2 The number 24 can be written as a product of two numbers in any of these ways.

$$\begin{array}{llll}
24 = 1\cdot 24 & \text{and} & 1+24=25 & \\
24 = -1\cdot(-24) & \text{and} & -1+(-24)=-25 & \\
24 = 2\cdot 12 & \text{and} & 2+12=14 & \\
24 = -2\cdot(-12) & \text{and} & -2+(-12)=-14 & \\
24 = 3\cdot 8 & \text{and} & 3+8=11 & \\
24 = -3\cdot(-8) & \text{and} & -3+(-8)=-11 & \leftarrow \quad \text{This is the correct choice.}\\
24 = 4\cdot 6 & \text{and} & 4+6=10 & \\
24 = -4\cdot(-6) & \text{and} & -4+(-6)=-10 &
\end{array}$$

Step 3 Re-write the middle term as $-11x = -3x - 8x$, so the problem becomes

$$6x^2 - 11x + 4 = 6x^2 - 3x - 8x + 4$$

Step 4 Factor by grouping. Factor a $3x$ from the first two terms and factor -4 from the last two terms.

$$3x(2x-1) - 4(2x-1)$$

Now factor the common binomial $(2x-1)$.

$$(2x-1)(3x-4)$$

Our answer is $(2x-1)(3x-4)$.

c) $5x^2 - 6x + 1$

Step 1 $ac = 5\cdot 1 = 5$

Step 2 The number 5 can be written as a product of two numbers in any of these ways.

$$\begin{array}{llll}
5 = 1\cdot 5 & \text{and} & 1+5=6 & \\
5 = -1\cdot(-5) & \text{and} & -1+(-5)=-6 & \leftarrow \quad \text{This is the correct choice}
\end{array}$$

Step 3 Rewrite the middle term as $-6x = -x - 5x$. The problem becomes

$$5x^2 - 6x + 1 = 5x^2 - x - 5x + 1$$

Step 4 Factor by grouping: factor an x from the first two terms and a factor of -1 from the last two terms

$$x(5x-1) - 1(5x-1)$$

Now factor the common binomial $(5x-1)$.

$$(5x-1)(x-1).$$

Our answer is $(5x-1)(x-1)$.

Solve Quadratic Equations by Factoring

Now that we know the basics of factoring, we can solve some simple polynomial equations. We already saw how we can use the Zero-product Property to solve polynomials in factored form. Here you will learn how to solve polynomials in expanded form. These are the steps for this process.

Step 1

If necessary, **re-write** the equation in standard form such that:

Polynomial expression $= 0$

Step 2

Factor the polynomial completely

Step 3

Use the zero-product rule to set **each factor equal to zero**

Step 4

Solve each equation from step 3

Step 5

Check your answers by substituting your solutions into the original equation

Example 7

Solve the following polynomial equations

a) $3x^2 - 24x + 36 = 0$

b) $2x^2 = 50$

c) $12x^2 - 7x - 10 = 0$

Solution:

a) $3x^2 - 24x + 36 = 0$

Rewrite this is not necessary since the equation is in the correct form.

Factor $3(x^2 - 8x + 12) = 0 \Rightarrow 3(x-6)(x-2) = 0$

Set each factor equal to zero.

$$x - 6 = 0 \qquad \text{or} \qquad x - 2 = 0$$

Solve

$$x = 6 \qquad \text{or} \qquad x = 2$$

Check Substitute each solution back into the original equation.

$$\begin{array}{llll} x = 6 & \Rightarrow & 3(6)^2 - 24(6) + 36 = 0 & \text{works out} \\ x = 2 & \Rightarrow & 3(2)^2 - 24(2) + 36 = 0 & \text{works out} \end{array}$$

Answer $x = 6, x = 2$

b) $2x^2 = 50$

Rewrite $2x^2 - 50 = 0$.

Factor $2(x^2 - 25) = 0 \Rightarrow 2(x+5)(x-5) = 0$.

Set each factor equal to zero:.

$$x - 5 = 0 \qquad \text{or} \qquad x + 5 = 0$$

Solve

$$x = 5 \qquad \text{or} \qquad x = -5$$

Check Substitute each solution back into the original equation.

$$x = 5 \Rightarrow 2(5)^2 - 50 = 0 \qquad \text{works out}$$
$$x = -5 \Rightarrow 2(-5)^2 - 50 = 0 \qquad \text{works out}$$

Answer $x = 5, x = -5$

c) $12x^2 - 7x - 10 = 0$

Rewrite No needed.

Factor

$$(3x+2)(4x-5)$$

.

Set each factor equal to zero:.

$$3x + 2 = 0 \qquad \text{or} \qquad 4x - 5 = 0$$

Solve

$$x = -\frac{2}{3} \qquad \text{or} \qquad x = \frac{5}{4}$$

Check Substitute each solution back into the original equation.

$$x = -2/3 \Rightarrow 12(-2/3)^2 - 7(-2/3) - 10 = 0 \qquad \text{works out}$$
$$x = 5/4 \Rightarrow 12(5/4)^2 - 7(5/4) - 10 = 0 \qquad \text{works out}$$

Answer $x = -\frac{2}{3}, x = \frac{5}{4}$

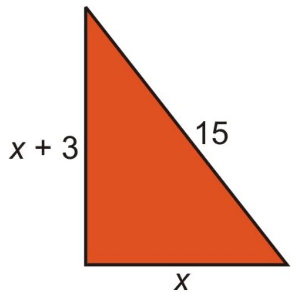
Solve Real-World Problems Using Polynomial Equations

Now that we know most of the factoring strategies for quadratic polynomials we can see how these methods apply to solving real world problems.

Example 8 Pythagorean Theorem

One leg of a right triangle is 3 feet longer than the other leg. The hypotenuse is 15 feet. Find the dimensions of the right triangle.

Solution



Let $x =$ the length of one leg of the triangle, then the other leg will measure $x+3$.

Lets draw a diagram.

Use the Pythagorean Theorem $(\text{leg}_1)^2 + (\text{leg}_2)^2 = (\text{hypotenuse})^2$ or $a^2+b^2=c^2$.

Here a and b are the lengths of the legs and c is the length of the hypotenuse.

Lets substitute the values from the diagram.

$$\begin{aligned} a^2+b^2 &= c^2 \\ x^2+(x+3)^2 &= 15^2 \end{aligned}$$

In order to solve, we need to get the polynomial in standard form. We must first distribute, collect like terms and **re-write** in the form polynomial $= 0$.

$$\begin{aligned} x^2+x^2+6x+9 &= 225 \\ 2x^2+6x+9 &= 225 \\ 2x^2+6x-216 &= 0 \end{aligned}$$

Factor the common monomial $2(x+3x-108)=0$.

To factor the trinomial inside the parenthesis we need to numbers that multiply to -108 and add to 3. It would take a long time to go through all the options so lets try some of the bigger factors.

$-108=-12\cdot$	and	$-12+9=-3$		
$-108=12\cdot(-9)$	and	$12+(-9)=3$	$\leftarrow$	This is the correct choice.

We factor as: $2(x-9)(x+12)=0$.

Set each term equal to zero and solve

$$x-9=0 \quad \text{or} \quad x+12=0$$
$$x=9 \qquad\qquad x=-12$$

It makes no sense to have a negative answer for the length of a side of the triangle, so the answer must be the following.

Answer $x=9$ for one leg, and $x+3=12$ for the other leg.

Check $9^2+12^2=81+144=225=15^2$ so the answer checks.

Example 9 Number Problems

The product of two positive numbers is 60. Find the two numbers if one of the numbers is 4 more than the other.

Solution

Let $x=$ one of the numbers and $x+4$ equals the other number.

The product of these two numbers equals 60. We can write the equation.

$$x(x+4)=60$$

In order to solve we must write the polynomial in standard form. Distribute, collect like terms and re-write in the form polynomial $=0$.

$$x^2+4x=60$$
$$x^2+4x-60=0$$

Factor by finding two numbers that multiply to -60 and add to 4. List some numbers that multiply to -60:

$-60=-4\cdot 15$	and	$-4+15=11$		
$-60=4\cdot(-15)$	and	$4+(-15)=-11$		
$-60=-5\cdot 12$	and	$-5+12=7$		
$-60=5\cdot(-12)$	and	$5+(-12)=-7$		
$-60=-6\cdot 10$	and	$-6+10=4$	$\leftarrow$	This is the correct choice
$-60=6\cdot(-10)$	and	$6+(-10)=-4$		

The expression factors as $(x+10)(x-6)=0$.

Set each term equal to zero and solve.

$$ath=x+10=0 \quad \text{or} \quad x-6=0$$
$$x=-10 \qquad\qquad x=6$$

Since we are looking for positive numbers, the answer must be the following.

Answer $x = 6$ for one number, and $x + 4 = 10$ for the other number.

Check $6 \cdot 10 = 60$ so the answer checks.

Example 10 Area of a rectangle

A rectangle has sides of $x + 5$ *and* $x - 3$*. What value of* x *gives and area of 48*?

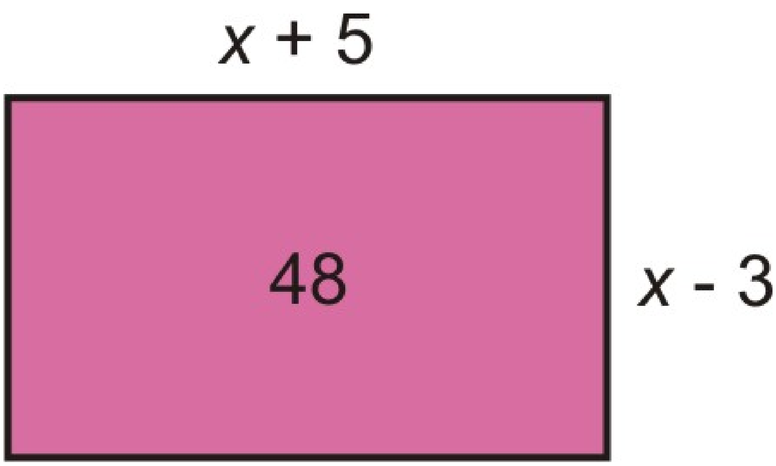


FIGURE 1.1

Solution:

Make a sketch of this situation.

Area of the rectangle $=$ length $\times$ width

$$ath = (x+5)(x-3) = 48$$

In order to solve, we must write the polynomial in standard form. Distribute, collect like terms and **rewrite** in the form polynomial $= 0$.

$$ath = x^2 + 2x - 15 = 48$$
$$x^2 + 2x - 63 = 0$$

Factor by finding two numbers that multiply to -63 and add to 2. List some numbers that multiply to -63.

$$ath = -63 = -7 \cdot 9 \quad \text{and} \quad -7 + 9 = 2 \quad \leftarrow \quad \text{This is the correct choice}$$
$$-63 = 7 \cdot (-9) \quad \text{and} \quad 7 + (-9) = -2$$

The expression factors as $(x+9)(x-7) = 0$.

Set each term equal to zero and solve.

$$ath = x + 9 = 0 \quad \text{or} \quad x - 7 = 0$$
$$x = -9 \qquad\qquad x = 7$$

Since we are looking for positive numbers the answer must be $x = 7$.

Answer The width is $x-3=4$ and the length is $x+5=12$.

Check $4 \cdot 12 = 48$ so the answer checks out.

Review Questions

Factor completely.

1. $2x^2+16x+30$
2. $-x^3+17x^2-70x$
3. $2x^2-512$
4. $12x^3+12x^2+3x$

Factor by grouping.

5. $6x^2-9x+10x-15$

6. $5x^2-35x+x-7$

7. $9x^2-9x-x+1$

8. $4x^2+32x-5x-40$

Factor the following quadratic binomials by grouping.

9. $4x^2+25x-21$

10. $6x^2+7x+1$

11. $4x^2+8x-5$

12. $3x^2+16x+21$

Solve.

13. $3x^2+24x+36=0$

14. $5x^2-45=0$

15. $20x^21-39x+18=0$

16. $4x^2+12x+9=0$

Solve the following application problems:

17. One leg of a right triangle is 7 feet longer than the other leg. The hypotenuse is 13 feet. Find the dimensions of the right triangle.

18. A rectangle has sides of $x+2$ and $x-1$. What value of x gives and area of 108?

19. The product of two positive numbers is 120. Find the two numbers if one numbers is 7 more than the other.

20. Framing Warehouse offers a picture framing service. The cost for framing a picture is made up of two parts. The cost of glass is \$1 per square foot. The cost of the frame is \$2 per linear foot. If the frame is a square, what size picture can you get framed for \$20?

Review Answers

1. $2(x+3)(x+5)$

2. $-x(x-7)(x-10)$
3. $2(x-4)(x+4)(x^2+16)$
4. $3x(2x+1)^2$
5. $(2x-3)(3x+5)$
6. $(x-7)(5x+1)$
7. $(9x-1)(x-1)$
8. $(x+8)(4x-5)$
9. $(4x-3)(x+7)$
10. $(6x+1)(x+1)$
11. $(2x-1)(2x+5)$
12. $(x+3)(3x+7)$
13. $x=6, x=2$
14. $x=3, x=-3$
15. $x=\frac{3}{4}, x=\frac{6}{5}$
16. $x=-\frac{3}{2}$
17. Leg $1=5$, Leg $2=12$
18. $x=10$
19. Numbers are 8 and 15.
20. You can frame a 2 foot $\times$ 2 foot picture.

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9619.

CHAPTER 2

Radical Equations and Radical Functions

Chapter Outline

2.1 Graphs of Square Root Functions

Learning Objectives

- Graph and compare square root functions.
- Shift graphs of square root functions.
- Graph square root functions using a graphing calculator.
- Solve real-world problems using square root functions.

Introduction

In this chapter, you will be learning about a different kind of function called the **square root function**. You have seen that taking the square root is very useful in solving quadratic equations. For example, to solve the equation $x^2 = 25$ we take the square root of both sides $\sqrt{x^2} = \pm\sqrt{25}$ and obtain $x = \pm 5$. A square root function has the form $y = \sqrt{f(x)}$. In this type of function, the expression in terms of x is found inside the square root sign (also called the "radical" sign).

Graph and Compare Square Root Functions

The square root function is the first time where you will have to consider the domain of the function before graphing. The domain is very important because the function is undefined if the expression inside the square root sign is negative, and as a result there will be no graph in that region.

In order to is cover how the graphs of square root function behave, we should make a table of values and plot the points.

Example 1

Graph the function $f(x) = \sqrt{x}$.

Solution

Before we make a table of values, we need to find the domain of this square root function. The domain is found by realizing that the function is only defined when the expression inside the square root is greater than or equal to zero. We find that the domain is all values of x such that $x \geq 0$.

This means that when we make our table of values, we should pick values of x that are greater than or equal to zero. It is very useful to include the value of zero as the first value in the table and include many values greater than zero. This will help us in determining what the shape of the curve will be. It is often helpful to replace f(x) with y to complete the table of values.

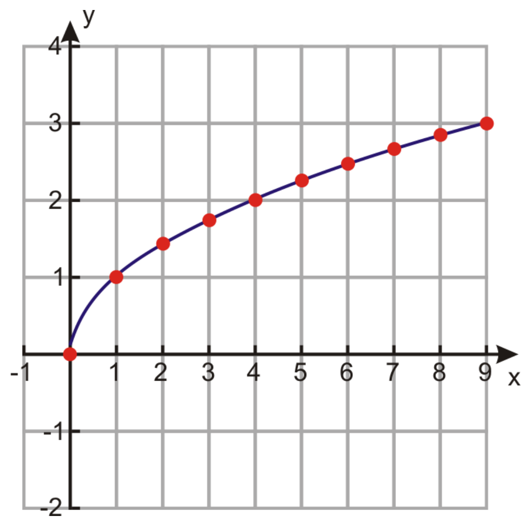
TABLE 2.1:

x	$y = \sqrt{x}$
0	$y = \sqrt{0} = 0$
1	$y = \sqrt{1} = 1$
2	$y = \sqrt{2} = 1.4$
3	$y = \sqrt{3} = 1.7$
4	$y = \sqrt{4} = 2$

TABLE 2.1: (continued)

x	$y = \sqrt{x}$
5	$y = \sqrt{5} = 2.2$
6	$y = \sqrt{6} = 22.4$
7	$y = \sqrt{7} = 2.6$
8	$y = \sqrt{8} = 2.8$
9	$y = \sqrt{9} = 3$

Here is what the graph of this table looks like.



The graphs of square root functions are always curved. The curve above looks like half of a parabola lying on its side. In fact the square root function we graphed above comes from the expression $y^2 = x$.

This is in the form of a parabola but with the x and y switched. We see that when we solve this expression for y we obtain two solutions $y = \sqrt{x}$ and $y = -\sqrt{x}$. The graph above shows the positive square root of this answer.

Example 2

Graph the function $f(x) = -\sqrt{x}$.

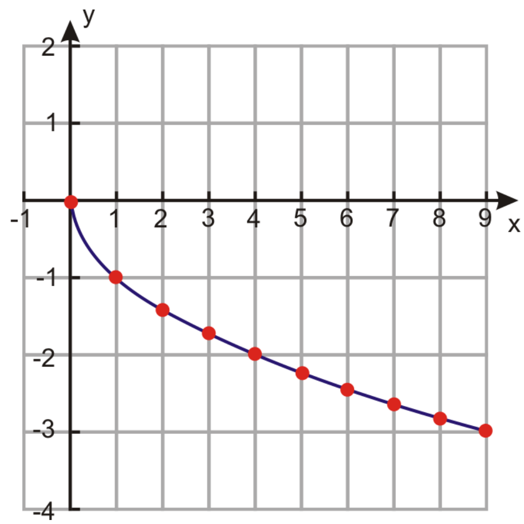
Solution

Once again, we must look at the domain of the function first. We see that the function is defined only for $x \geq 0$. Let's make a table of values and calculate a few values of the function.

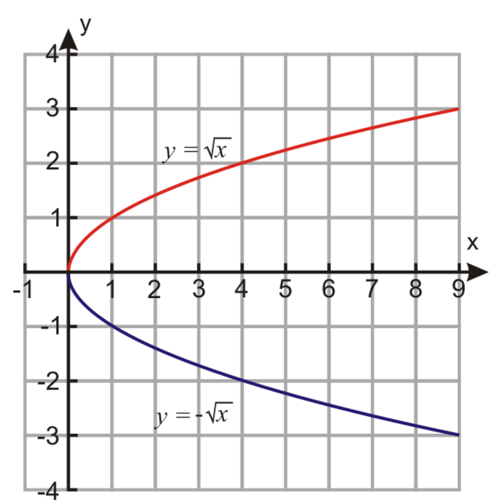
TABLE 2.2:

x	$y = -\sqrt{x}$
0	$y = -\sqrt{0} = -0$
1	$y = -\sqrt{1} = -1$
2	$y = -\sqrt{2} = -1.4$
3	$y = -\sqrt{3} = -1.7$
4	$y = -\sqrt{4} = -2$
5	$y = -\sqrt{5} = -2.2$
6	$y = -\sqrt{6} = -22.4$
7	$y = -\sqrt{7} = -2.6$
8	$y = -\sqrt{8} = 2.8$
9	$y = -\sqrt{9} = -3$

Here is the graph from this table.



Notice that if we graph the two separate functions on the same coordinate axes, the combined graph is a parabola lying on its side.



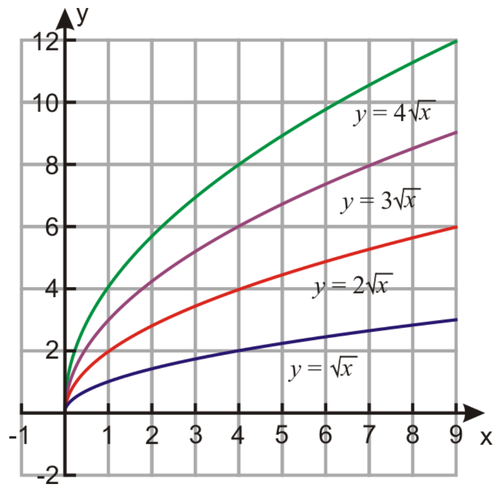
Now let's compare square root functions that are multiples of each other.

Example 3

Graph functions $f(x) = \sqrt{x}, f(x) = 2\sqrt{x}, f(x) = 3\sqrt{x}, f(x) = 4\sqrt{x}$ *on the same graph.*

Solution

Here we will show just the graph without the table of values.



If we multiply the function by a constant bigger than one, the function increases faster the greater the constant is.

Example 4

Graph functions $a. f(x) = \sqrt{x}, b. f(x) = \sqrt{2x}, c. f(x) = \sqrt{3x}, d. f(x) = \sqrt{4x}$ *on the same graph.*

Solution

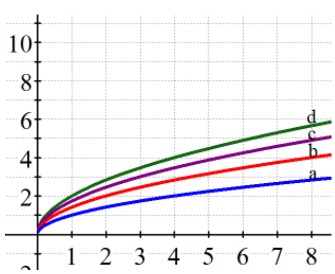


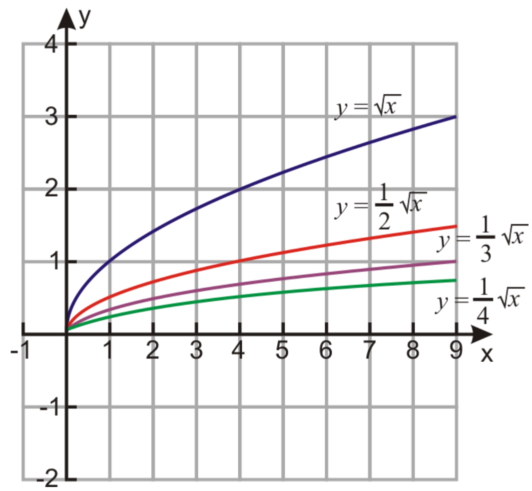
FIGURE 2.1

Notice that multiplying the expression inside the square root by a constant has the same effect as in the previous example but the function increases at a slower rate because the entire function is effectively multiplied by the square root of the constant. Also note that the graph of $\sqrt{4x}$ is the same as the graph of $2\sqrt{2x}$. This makes sense algebraically since $\sqrt{4} = 2$.

Example 5

Graph functions $f(x) = \sqrt{x}, f(x) = \frac{1}{2}\sqrt{x}, f(x) = \frac{1}{3}\sqrt{x}, f(x) = \frac{1}{4}\sqrt{x}$ *on the same graph.*

Solution



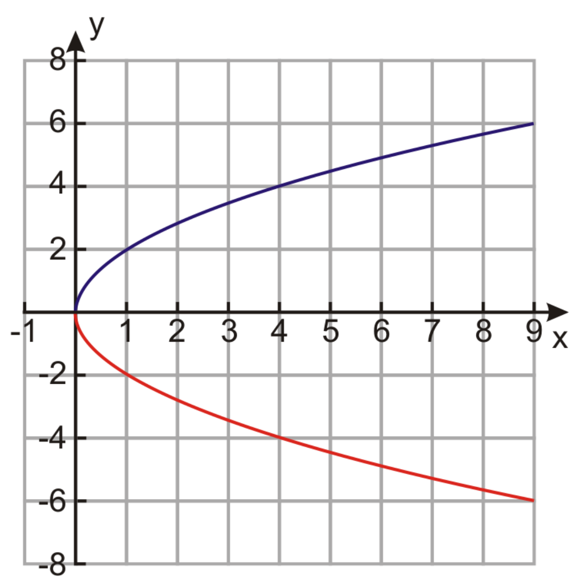
If we multiply the function by a constant between 0 and 1, the function increases at a slower rate for smaller constants.

Example 6

Graph functions $f(x) = 2\sqrt{x}, f(x) = -2\sqrt{x}$ *on the same graph.*

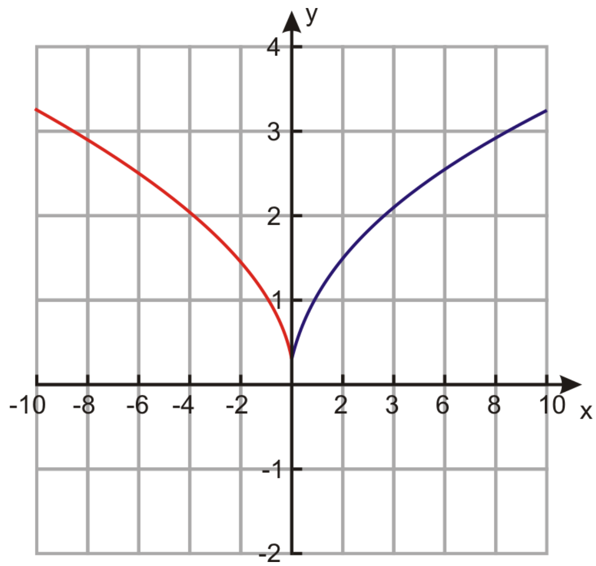
Solution

If we multiply the function by a negative function, the square root function is reflected about the $x-$axis.



Example 7

Graph functions $f(x) = \sqrt{x}, f(x) = \sqrt{-x}$ *on the same graph.*



Solution

Notice that for function $f(x) = \sqrt{x}$ the domain is values of $x \geq 0$, and for function $f(x) = \sqrt{-x}$ the domain is values of $x \leq 0$.

When we multiply the argument of the function by a negative constant the function is reflected about the $y-$axis.

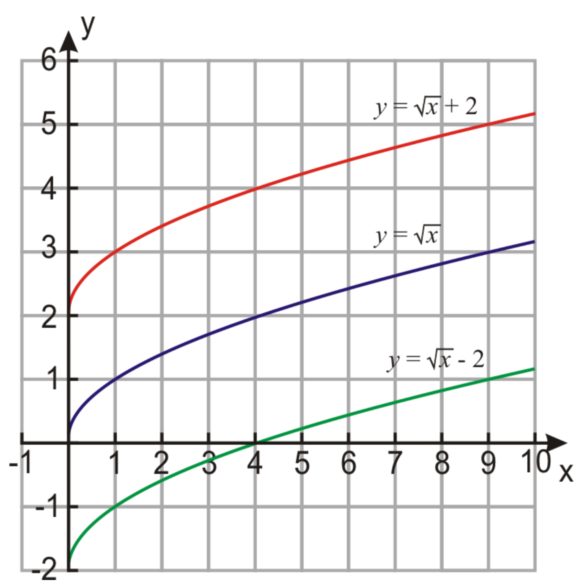
Shift Graphs of Square Root Functions

Now, let's see what happens to the square root function as we add positive and negative constants to the function.

Example 8

Graph the functions $f(x) = \sqrt{x}, f(x) = \sqrt{x}+2, f(x) = \sqrt{x}-2$.

Solution

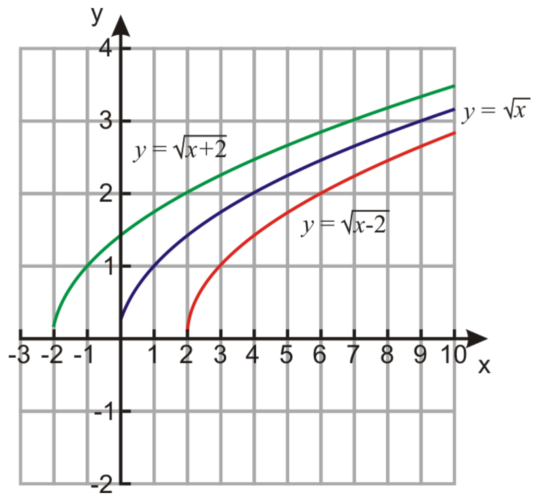


We see that the graph keeps the same shape, but moves up for positive constants and moves down for negative constants.

Example 9

Graph the functions $f(x) = \sqrt{x}, f(x) = \sqrt{x-2}, f(x) = \sqrt{x+2}$.

Solution



When we add constants to the argument of the function, the function shifts to the left for a positive constant and to the right for a negative constant because the domain shifts. There can't be a negative number inside the square root.

Now let's graph a few more examples of square root functions.

Example 10

Graph the function $f(x) = 2\sqrt{3x-1}+2$.

Solution

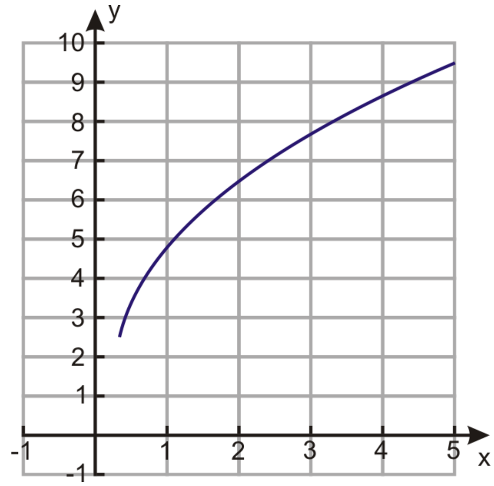
We first determine the domain of the function. The function is only defined if the expression inside the square root is positive $3x-1 \geq 0$ or $x \geq \frac{1}{3}$.

Make a table for values of x greater than or equal to $\frac{1}{3}$.

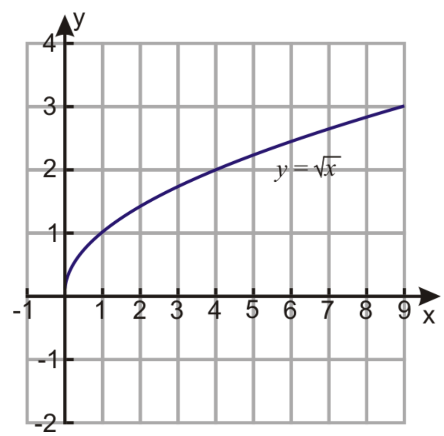
TABLE 2.3:

x	$y = 2\sqrt{3x-1}+2$
$\frac{1}{3}$	$y = 2\sqrt{3 \cdot \frac{1}{3} - 1} + 2 = 2$
1	$y = 2\sqrt{3(1)-1}+2 = 4.8$
2	$y = 2\sqrt{3(2)-1}+2 = 6.5$
3	$y = 2\sqrt{3(3)-1}+2 = 7.7$
4	$y = 2\sqrt{3(4)-1}+2 = 8.6$
5	$y = 2\sqrt{3(5)-1}+2 = 9.5$

Here is the graph.

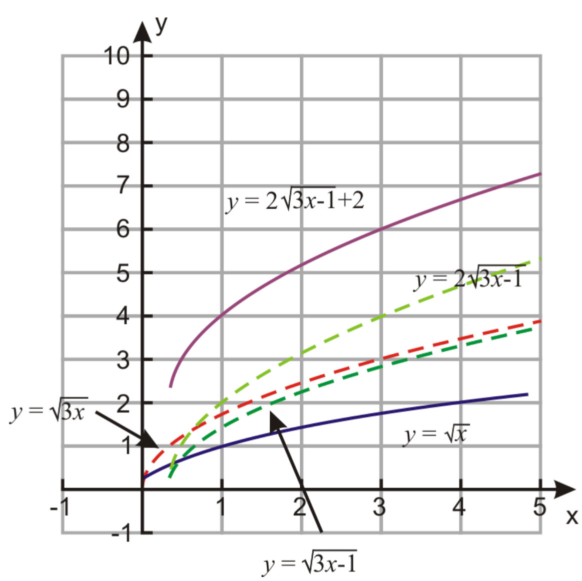


You can also think of this function as a combination of shifts and stretches of the basic square root function $y = \sqrt{x}$. We know that the graph of this function looks like the one below.



If we multiply the argument by 3 to obtain $y = \sqrt{3x}$, this stretches the curve vertically because the value of y increases faster by a factor of $y = \sqrt{3}$.

Next, when we subtract the value of 1 from the argument to obtain $y = \sqrt{3x-1}$ this shifts the entire graph to the left by one unit.



Multiplying the function by a factor of 2 to obtain $y = 2\sqrt{3x-1}$ stretches the curve vertically again, and y increases faster by a factor of 2.

Finally, we add the value of 2 to the function to obtain $y = \sqrt{3x-1}+2$. This shifts the entire function vertically by 2 units.

This last method of graphing showed a way to graph functions without making a table of values. If we know what the basic function looks like, we can use shifts and stretches to **transform** the function and get to the desired result.

Graph Square Root Functions Using a Graphing Calculator

Next, we will demonstrate how to use the graphing calculator to plot square root functions.

Example 11

Graph the following functions using a graphing calculator.

a) $f(x) = \sqrt{x+5}$

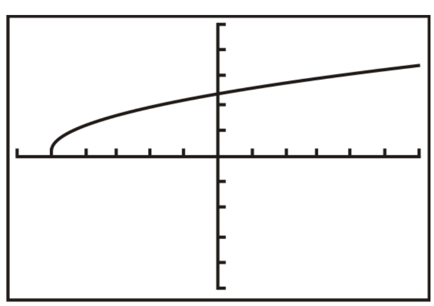
b) $f(x) = \sqrt{9-x^2}$

Solution:

In all the cases we start by pressing the **[Y=button]** and entering the function on the function screen of the calculator:

We then press **[GRAPH]** to display the results. Make sure your window is set appropriately in order to view the function well. This is done by pressing the **[WINDOW]** button and choosing appropriate values for the Xmin, Xmax, Ymin and Ymax.

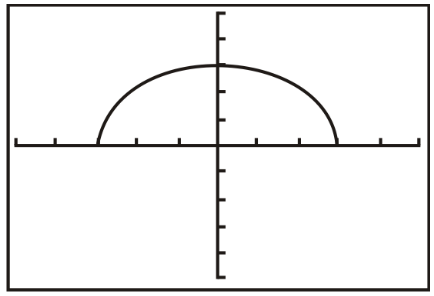
a)



The window of this graph is $-6 \le x \le 5$; $-5 \le y \le 5$.

The domain of the function is $x \ge -5$

b)

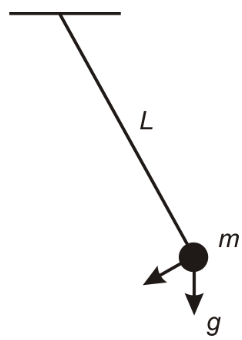


The window of this graph is $-5 \leq x \leq 5; -5 \leq y \leq 5$.

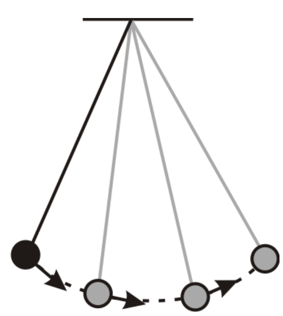
The domain of the function is $-3 \leq x \leq 3$

Solve Real-World Problems Using Square Root Functions

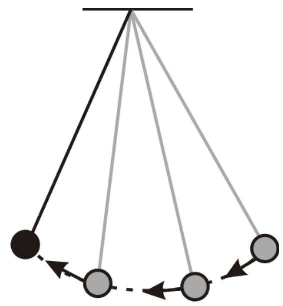
Pendulum



Mathematicians and physicists have studied the motion of a pendulum in great detail because this motion explains many other behaviors that occur in nature. This type of motion is called **simple harmonic motion** and it is very important because it describes anything that repeats periodically. Galileo was the first person to study the motion of a pendulum around the year 1600. He found that the time it takes a pendulum to complete a swing from a starting point back to the beginning does not depend on its mass or on its angle of swing (as long as the angle of the swing is small). Rather, it depends only on the length of the pendulum.



The time it takes a pendulum to swing from a starting point and back to the beginning is called the **period** of the pendulum.



Galileo found that the period of a pendulum is proportional to the square root of its length $T = a\sqrt{L}$. The proportionality constant depends on the acceleration of gravity $a = \frac{2\pi}{\sqrt{g}}$. At sea level on Earth the acceleration of gravity is $g = 9.81\ m/s^2$ (meters per second squared). Using this value of gravity, we find $a = 2.0$ with units of $\frac{s}{\sqrt{m}}$ (seconds divided by the square root of meters). Up until the mid 20th century, all clocks used pendulums as their central time keeping component.

Example 12

Graph the period of a pendulum of a clock swinging in a house on Earth at sea level as we change the length of the pendulum. What does the length of the pendulum need to be for its period to be one second?

Solution

The function for the period of a pendulum at sea level is: $T = 2\sqrt{L}$.

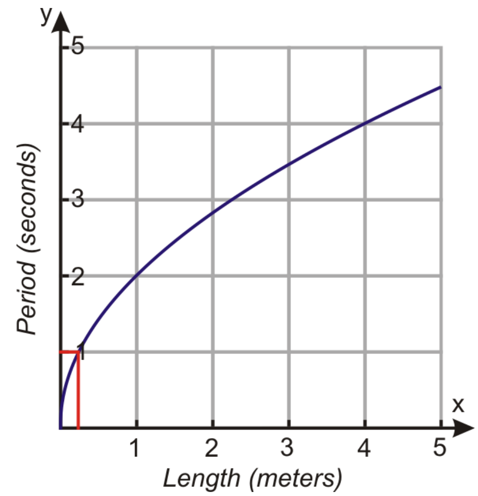
We make a graph with the horizontal axis representing the length of the pendulum and with the vertical axis representing the period of the pendulum.

We start by making a table of values.

TABLE 2.4:

L	$T = 2\sqrt{L}$
0	$T = 2\sqrt{0} = 0$
1	$T = 2\sqrt{1} = 2$
2	$T = 2\sqrt{2} = 2.8$
3	$T = 2\sqrt{3} = 3.5$
4	$T = 2\sqrt{4} = 4$
5	$T = 2\sqrt{5} = 4.5$

Now let's graph the function.



We can see from the graph that a length of approximately $\frac{1}{4}$ meters gives a period of one second. We can confirm this answer by using our function for the period and plugging in $T = 1\ second$.

$$T = 2\sqrt{L} \Rightarrow 1 = 2\sqrt{L}$$

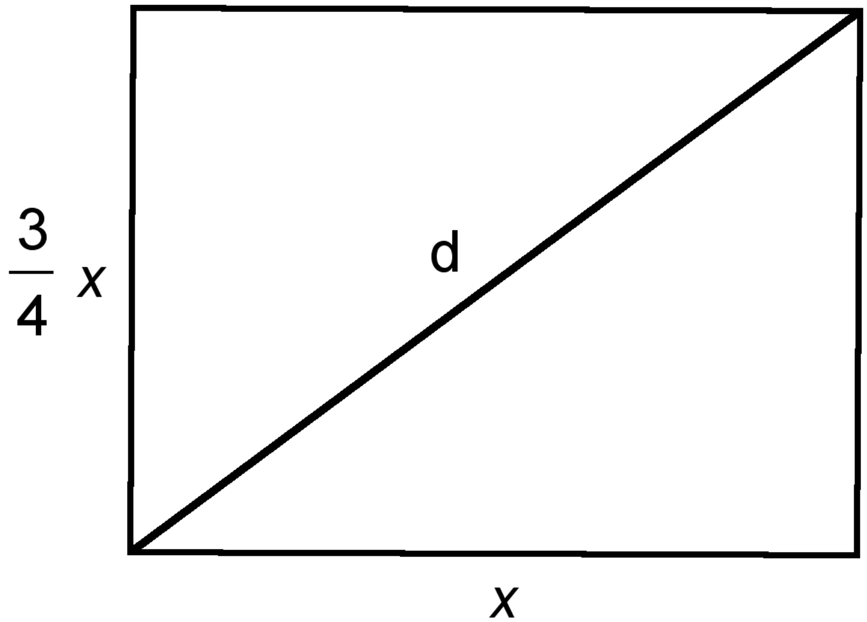
$$\text{Square both sides of the equation:} \qquad 1 = 4L$$

$$\text{Solve for } L: \qquad L = \frac{1}{4}\ meters$$

Example 13

"Square" TV screens have an aspect ratio of 4:3. This means that for every four inches of length on the horizontal, there are three inches of length on the vertical. TV sizes represent the length of the diagonal of the television screen. Graph the length of the diagonal of a screen as a function of the area of the screen. What is the diagonal of a screen with an area of $180\ in^2$*?*

Solution



Let $d =$ length of the diagonal, $x =$ horizontal length

$$4 \cdot \text{vertical length} = 3 \cdot \text{horizontal length}$$
$$\text{Or,}$$
$$\text{vertical length} = \frac{3}{4}x.$$

The area of the screen is: $A = \text{length} \cdot \text{width}$ or $A = \frac{3}{4}x^2$

Find how the diagonal length and the horizontal length are related by using the Pythagorean theorem, $a^2 + b^2 = c^2$.

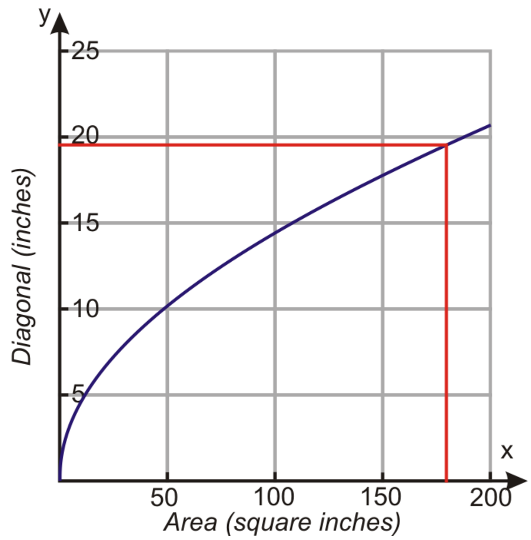
$$\begin{aligned}
x^2 + \left(\frac{3}{4}x\right)^2 &= d^2 \\
x^2 + \frac{9}{16}x^2 &= d^2 \\
\frac{25}{16}x^2 = d^2 \Rightarrow x^2 &= \frac{16}{25}d^2 \Rightarrow x^2 = \frac{4}{5}d \\
A = \frac{3}{4}\left(\frac{4}{5}d\right)^2 &= \frac{3}{4} \cdot \frac{16}{25}d^2 = \frac{12}{25}d^2
\end{aligned}$$

We can also find the diagonal length as a function of the area $d^2 = \frac{25}{12}A$ or $d = \frac{5}{2\sqrt{3}}\sqrt{A}$.

Make a graph where the horizontal axis represents the area of the television screen and the vertical axis is the length of the diagonal. Let's make a table of values.

TABLE 2.5:

A	$d = \frac{5}{2\sqrt{3}}\sqrt{A}$
0	0
25	7.2
50	10.2
75	12.5
100	14.4
125	16.1
150	17.6
175	19
200	20.4



From the graph we can estimate that when the area of a TV screen is 180 in^2 the length of the diagonal is approximately 19.5 inches inches. We can confirm this by substituting $a = 180$ into the formula that relates the diagonal to the area.

$$d = \frac{5}{2\sqrt{3}}\sqrt{A} = \frac{5}{2\sqrt{3}}\sqrt{180} = 19.4 \textit{ inches}$$

Review Questions

Graph the following functions on the same coordinate axes.

1. $f(x) = \sqrt{x}, f(x) = 2.5\sqrt{x}$ and $f(x) = -2.5\sqrt{x}$
2. $f(x) = \sqrt{x}, f(x) = 0.3\sqrt{x}$ and $f(x) = 0.6\sqrt{x}$
3. $f(x) = \sqrt{x}, f(x) = \sqrt{x-5}$ and $f(x) = \sqrt{x+5}$
4. $f(x) = \sqrt{x}, f(x) = \sqrt{x}+8$ and $f(x) = \sqrt{x}-8$

Graph the following functions.

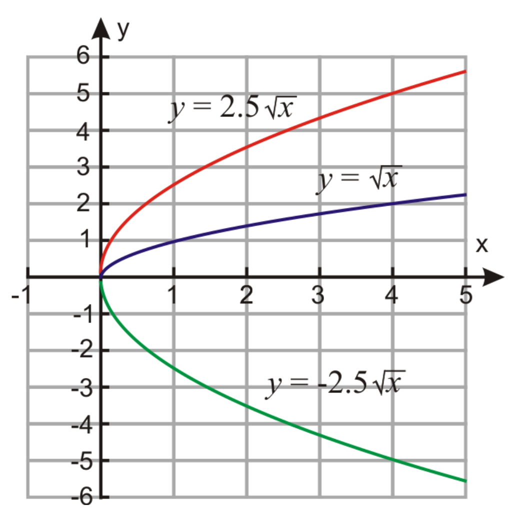
1. $f(x) = \sqrt{2x-1}$
2. $f(x) = \sqrt{4x+4}$

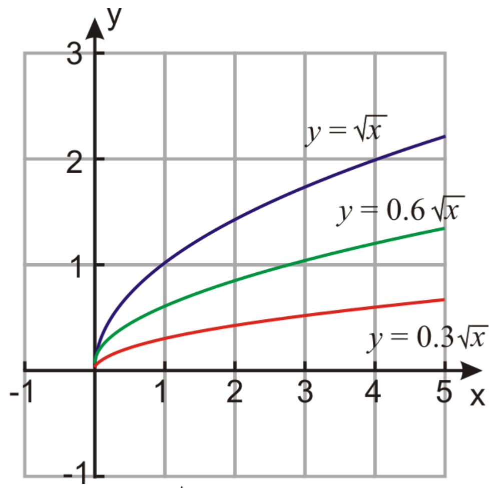
3. $f(x) = \sqrt{5-x}$
4. $f(x) = 2\sqrt{x}+5$
5. $f(x) = 3-\sqrt{x}$
6. $f(x) = 4+2\sqrt{x}$
7. $f(x) = 2\sqrt{2x+3}+1$
8. $f(x) = 4+2\sqrt{2-x}$
9. $f(x) = \sqrt{x+1}-\sqrt{4x-5}$
10. The acceleration of gravity can also given in feet per second squared. It is $g = 32\ ft/s^2$ at sea level. Graph the period of a pendulum with respect to its length in feet. For what length in feet will the period of a pendulum be two seconds?
11. The acceleration of gravity on the Moon is $1.6\ m/s^2$. Graph the period of a pendulum on the Moon with respect to its length in meters. For what length, in meters, will the period of a pendulum be 10 seconds?
12. The acceleration of gravity on Mars is $3.69\ m/s^2$. Graph the period of a pendulum on the Mars with respect to its length in meters. For what length, in meters, will the period of a pendulum be three seconds?
13. The acceleration of gravity on the Earth depends on the latitude and altitude of a place. The value of g is slightly smaller for places closer to the Equator than places closer to the Poles, and the value of g is slightly smaller for places at higher altitudes that it is for places at lower altitudes. In Helsinki, the value of $g = 9.819\ m/s^2$, in Los Angeles the value of $g = 9.796\ m/s^2$ and in Mexico City the value of $g = 9.779\ m/s^2$. Graph the period of a pendulum with respect to its length for all three cities on the same graph. Use the formula to find the length (in meters) of a pendulum with a period of 8 seconds for each of these cities.
14. The aspect ratio of a wide-screen TV is 2.39:1. Graph the length of the diagonal of a screen as a function of the area of the screen. What is the diagonal of a screen with area 150 in^2?

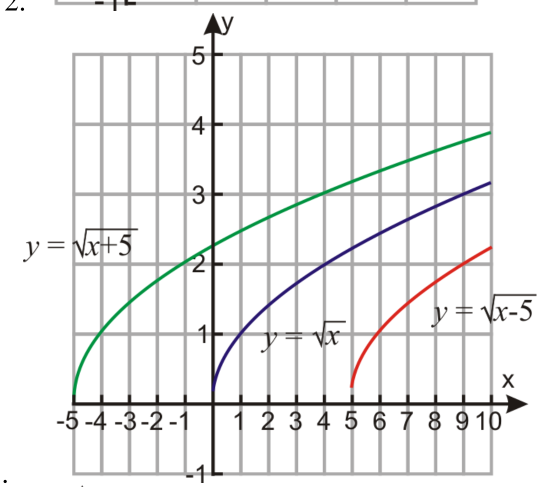
Graph the following functions using a graphing calculator.

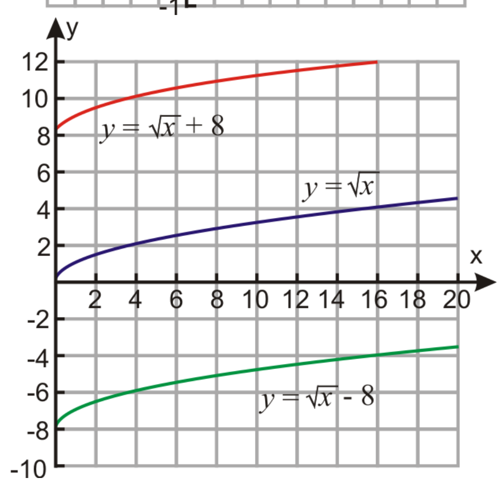
1. $f(x) = \sqrt{3x-2}$
2. $f(x) = 4+\sqrt{2-x}$
3. $f(x) = \sqrt{x^2-9}$
4. $f(x) = \sqrt{x}-\sqrt{x+2}$

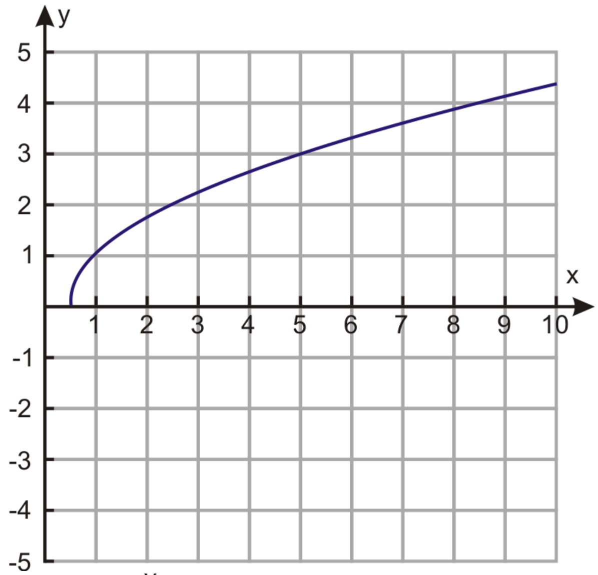
Review Answers

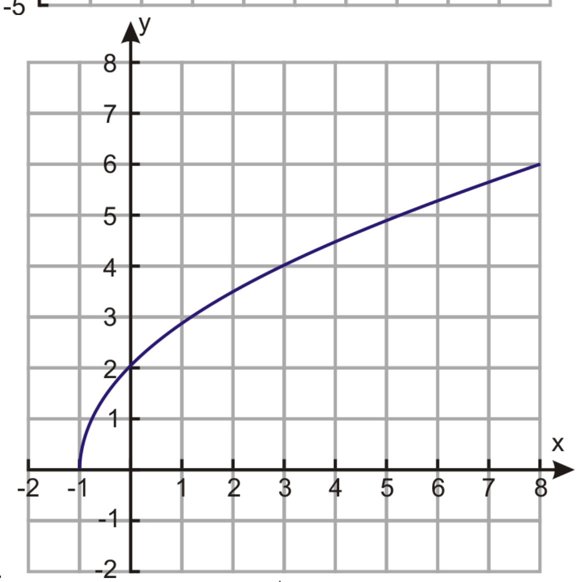
1. 

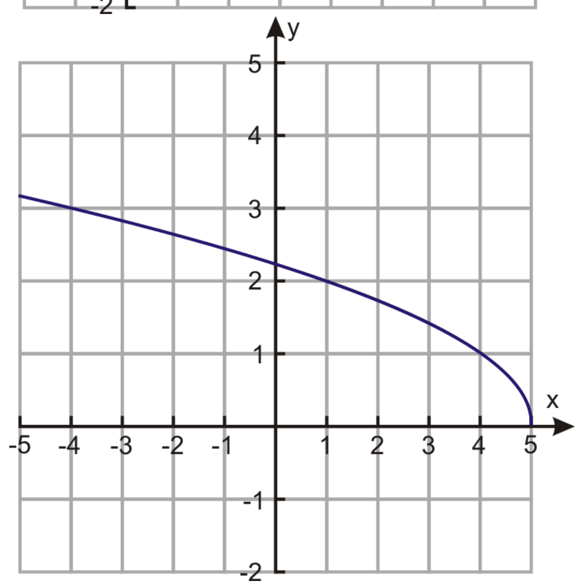
2. 
y
x
3
2
1
-1
-1
1
2
3
4
5
y = √x
y = 0.6 √x
y = 0.3√x

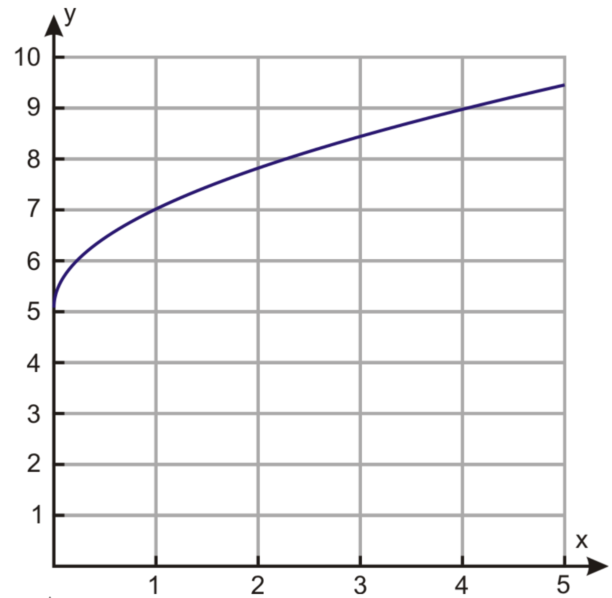
3. 
y
x
5
4
3
2
1
-1
-5 -4 -3 -2 -1
1 2 3 4 5 6 7 8 9 10
y = √x+5
y = √x
y = √x-5

4. 
y
x
12
10
8
6
4
2
-2
-4
-6
-8
-10
2 4 6 8 10 12 14 16 18 20
y = √x + 8
y = √x
y = √x - 8

5. 
y
5
4
3
2
1
x
1 2 3 4 5 6 7 8 9 10
-1
-2
-3
-4
-5

6. 
y
8
7
6
5
4
3
2
1
x
-2 -1 1 2 3 4 5 6 7 8
-1
-2

7. 
y
5
4
3
2
1
x
-5 -4 -3 -2 -1 1 2 3 4 5
-1
-2

8. 
y
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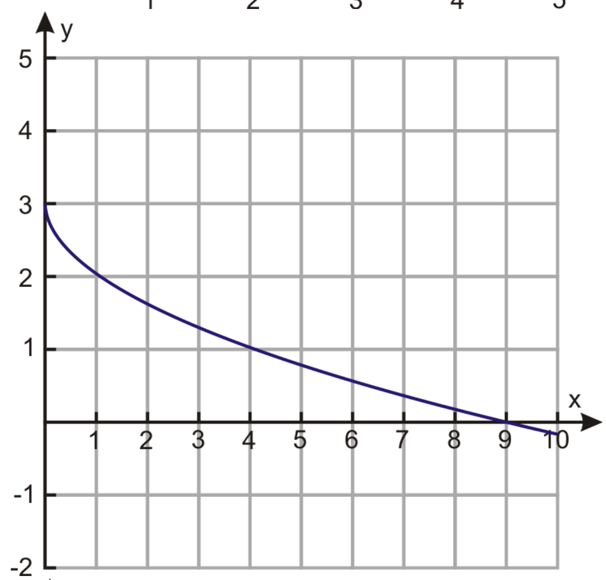
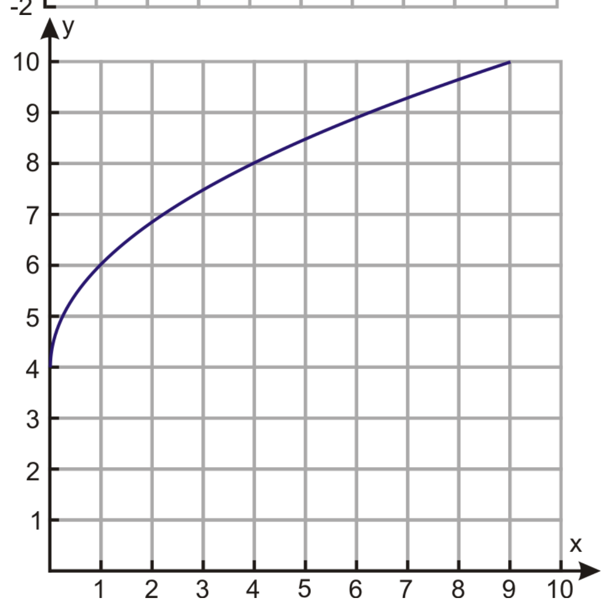
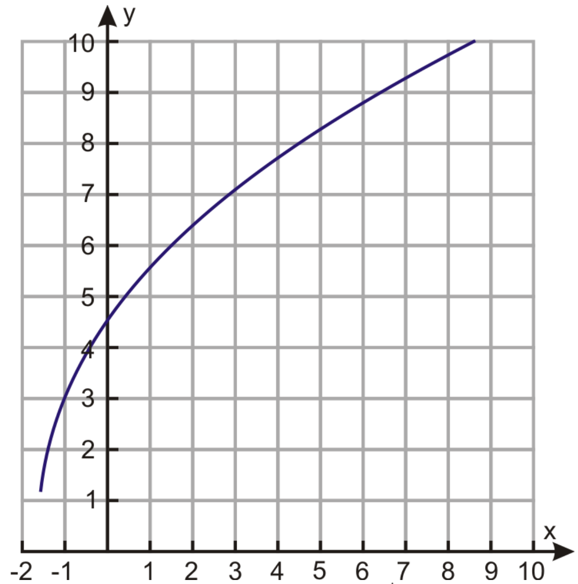
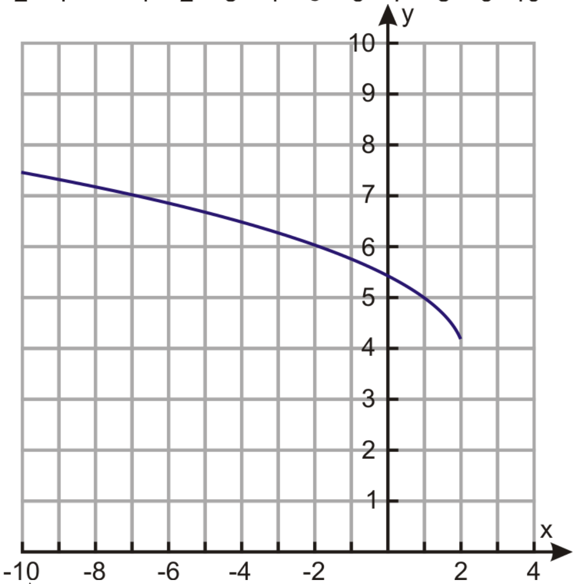
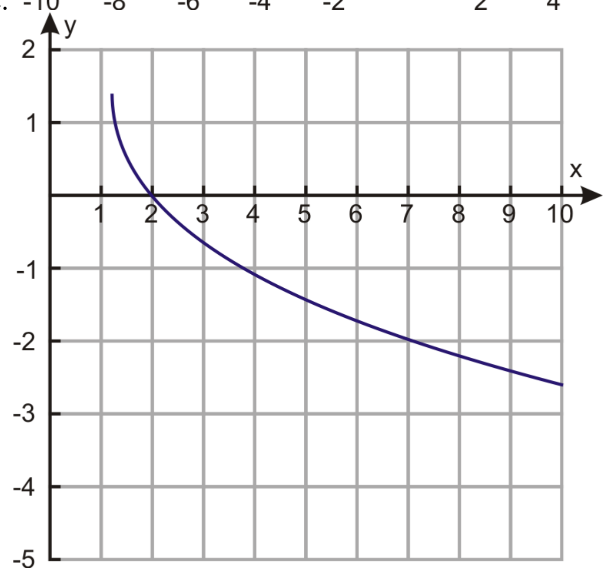
9
8
7
6
5
4
3
2
1
x
1
2
3
4
5

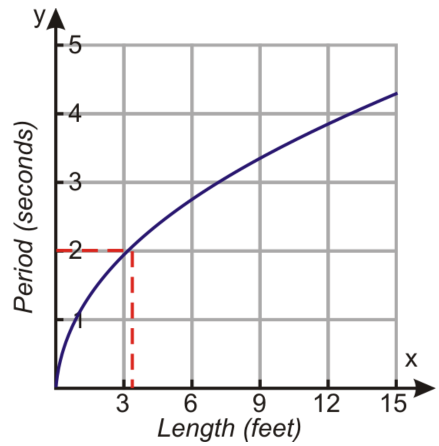
9. 
y
5
4
3
2
1
x
1 2 3 4 5 6 7 8 9 10
-1
-2

10. 
y
10
9
8
7
6
5
4
3
2
1
x
1 2 3 4 5 6 7 8 9 10

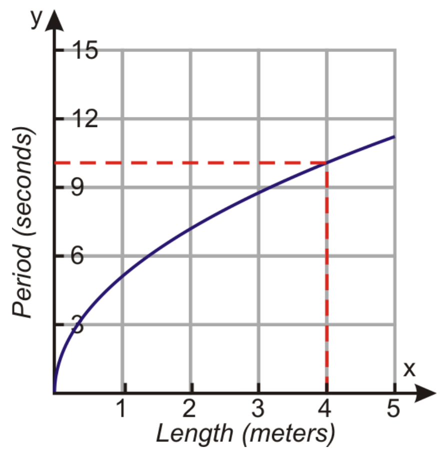
11. 
y
x
-2 -1 1 2 3 4 5 6 7 8 9 10
1 2 3 4 5 6 7 8 9 10

12. 
y
x
-10 -8 -6 -4 -2 2 4
1 2 3 4 5 6 7 8 9 10

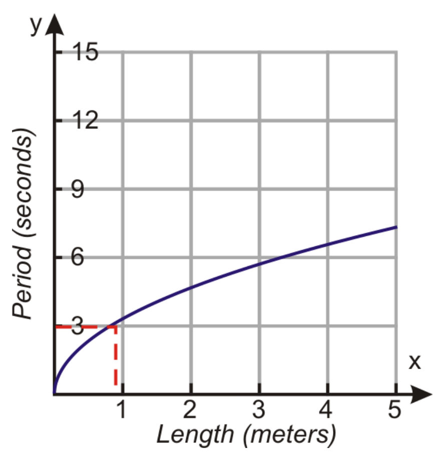
13. 
y
x
1 2 3 4 5 6 7 8 9 10
2 1 -1 -2 -3 -4 -5

14. 

$L = 3.25\ feet$

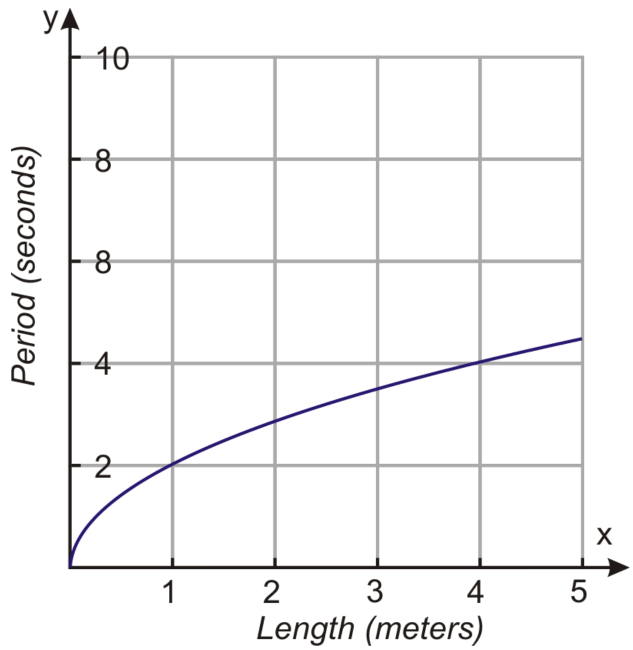
15. 

$L = 4.05\ meters$

16. 

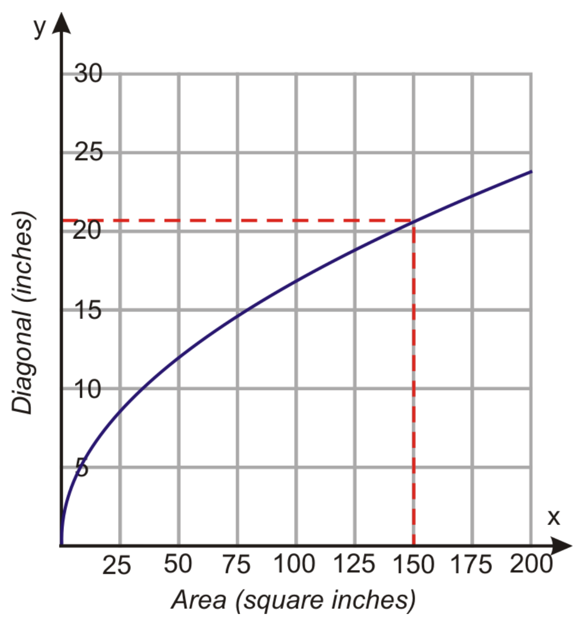
$L = 0.84\ meters$

17.



Note: The differences are so small that all of the lines appear to coincide on this graph. If you zoom (way) in you can see slight differences. The period of an 8 meter pedulum in Helsinki is 1.8099 seconds, in Los Angeles it is 1.8142 seconds, and in Mexico City it is 1.8173 seconds.

1.



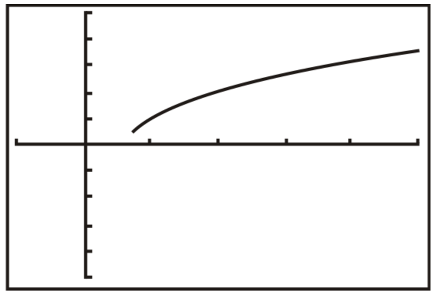
$D = 20.5$ *inches*

15.92 m Helsinki

15.88 m Los Angeles

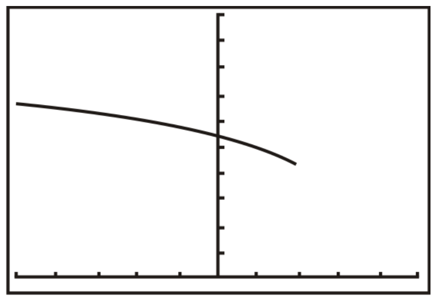
15.85 m Mexico City

19.



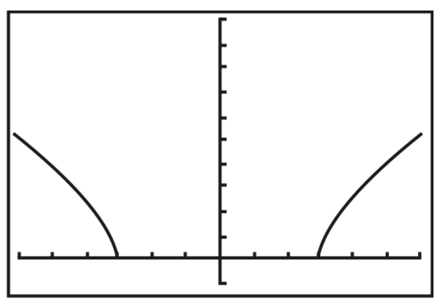
Window $-1 \leq x \leq 5; -5 \leq y \leq 5$

20.



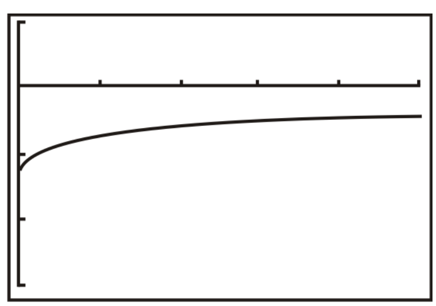
Window $-5 \leq x \leq 5; 0 \leq y \leq 10$

21.



Window $-6 \leq x \leq 6; -1 \leq y \leq 10$

22.



$0 \leq x \leq 5; -3 \leq y \leq 1$

2.2 Radical Expressions I

Learning objectives

- Use the product and quotient properties of radicals to simplify radicals.
- Add and subtract radical expressions.
- Solve real-world problems using square root functions.

Introduction

A radical reverses the operation of raising a number to a power. For example, to find the square of 4 we write $4^2 = 4 \cdot 4 = 16$. The reverse process is called finding the square root. The symbol for a square root is $\sqrt{\ }$. This symbol is also called the **radical sign**. When we take the square root of a number, the result is a number which when squared gives the number under the square root sign. For example,

$$\sqrt{9} = 3 \qquad \text{since} \qquad 3^2 = 3 \cdot 3 = 9$$

Radicals often have an index in the top left corner. The index indicates which root of the number we are seeking. Square roots have an index of 2 but many times this index is not written.

$$\sqrt[2]{36} = 6 \qquad \text{since} \qquad 6^2 = 36$$

The cube root of a number gives a number which when raised to the third power gives the number under the radical sign.

$$\sqrt[3]{64} = 4 \qquad \text{since} \qquad 4^3 = 4 \cdot 4 \cdot 4 = 64$$

The fourth root of number gives a number which when raised to the power four gives the number under the radical sign.

$$\sqrt[4]{81} = 3 \qquad \text{since} \qquad 3^4 = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

Even and odd roots

Radical expressions that have even indices are called **even roots** and radical expressions that have odd indices are called **odd roots**. There is a very important difference between even and odd roots in that they give drastically different results when the number inside the radical sign is negative.

Any real number raised to an even power results in a positive answer. Therefore, when the index of a radical is even, the number inside the radical sign must be non-negative in order to get a real answer.

On the other hand, a positive number raised to an odd power is positive and a negative number raised to an odd power is negative. Thus, a negative number inside the radical sign is not a problem. It just results in a negative answer.

Example 1

Evaluate each radical expression.

a) $\sqrt{121}$

b) $\sqrt[3]{125}$

c) $\sqrt[4]{-625}$

d) $\sqrt[5]{-32}$

Solution

a) $\sqrt{121} = 11$

b) $\sqrt[3]{125} = 5$

c) $\sqrt[4]{-625}$ is not a real number

d) $\sqrt[5]{-32} = -2$

Use the Product and Quotient Properties of Radicals

Radicals can be rewritten as exponent with rational powers. The radical $y = \sqrt[m]{a^n}$ is defined as $a^{\frac{n}{m}}$.

Example 2

Write each expression as an exponent with a rational value for the exponent.

a) $\sqrt{5}$

b) $\sqrt[4]{a}$

c) $\sqrt[3]{4xy}$

d) $\sqrt[6]{x^5}$

Solution

a) $\sqrt{5} = 5^{\frac{1}{2}}$

b) $\sqrt[4]{a} = a^{\frac{1}{4}}$

c) $\sqrt[3]{4xy} = (4xy)^{\frac{1}{3}}$

d) $\sqrt[6]{x^5} = x^{\frac{5}{6}}$

As a result of this property, for any non-negative number $\sqrt[n]{a^n} = a^{\frac{n}{n}} = a$.

Since roots of numbers can be treated as powers, we can use exponent rules to simplify and evaluate radical expressions. Let's review the product and quotient rule of exponents.

Raising a product to a power	$(x \cdot y)^n = x^n \cdot y^n$
Raising a quotient to a power	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$

In radical notation, these properties are written as

Raising a product to a power	$\sqrt[m]{x \cdot y} = \sqrt[m]{x} \cdot \sqrt[m]{y}$
Raising a quotient to a power	$\sqrt[m]{\frac{x}{y}} = \frac{\sqrt[m]{x}}{\sqrt[m]{y}}$

A very important application of these rules is reducing a radical expression to its simplest form. This means that we apply the root on all the factors of the number that are perfect roots and leave all factors that are not perfect roots inside the radical sign.

For example, in the expression $\sqrt{16}$, the number is a perfect square because $16 = 4^2$. This means that we can simplify.

$$\sqrt{16} = \sqrt{4^2} = 4$$

Thus, the square root disappears completely.

On the other hand, in the expression, the number $\sqrt{32}$ is not a perfect square so we cannot remove the square root. However, we notice that $32 = 16 \cdot 2$, so we can write 32 as the product of a perfect square and another number.

$$\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16 \cdot 2}$$

If we apply the "raising a product to a power" rule we obtain

$$\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16} \cdot \sqrt{2}$$

Since $\sqrt{16} = 4$, we get $\sqrt{32} = 4 \cdot \sqrt{2} = 4\sqrt{2}$.

Example 3

Write the following expression in the simplest radical form.

a) $\sqrt{8}$

b) $\sqrt{50}$

c) $\sqrt{\frac{125}{72}}$

Solution

The strategy is to write the number under the square root as the product of a perfect square and another number. The goal is to find the highest perfect square possible, however, if we don't we can repeat the procedure until we cannot simplify any longer.

a) We can write $8 = 4 \cdot 2$ so $\sqrt{8} = \sqrt{4 \cdot 2}$

Use the rule for raising a product to a power $\sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2}$

Finally we have, $\sqrt{8} = 2\sqrt{2}$.

b) We can write $50 = 25 \cdot 2$ so $\sqrt{50} = \sqrt{25 \cdot 2}$

Use the rule for raising a product to a power $= \sqrt{25} \cdot \sqrt{2} = \underline{\underline{5\sqrt{2}}}$

c) Use the rule for raising a product to a power to separate the fraction.

$$\sqrt{\frac{125}{72}} = \frac{\sqrt{125}}{\sqrt{72}}$$

Rewrite each radical as a product of a perfect square and another number.

$$= \frac{\sqrt{25 \cdot 5}}{9 \cdot 6} = \frac{5\sqrt{5}}{3\sqrt{6}}$$

The same method can be applied to reduce radicals of different indices to their simplest form.

Example 4

Write the following expression in the simplest radical form.

a) $\sqrt{40}$

b) $\sqrt[4]{\frac{162}{80}}$

c) $\sqrt[3]{135}$

Solution

In these cases we look for the highest possible perfect cube, fourth power, etc. as indicated by the index of the radical.

a) Here we are looking for the product of the highest perfect cube and another number. We write

$$\sqrt[3]{40} = \sqrt[3]{8 \cdot 5} = 2\sqrt[3]{5}$$

b) Here we are looking for the product of the highest perfect fourth power and another number.

Rewrite as the quotient of two radicals $\quad \sqrt[4]{\frac{162}{80}} = \frac{\sqrt[4]{162}}{\sqrt[4]{80}}$

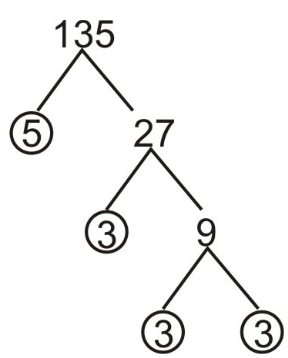
Simplify each radical separately $\quad = \frac{\sqrt[4]{81 \cdot 2}}{\sqrt[4]{16 \cdot 5}} = \frac{\sqrt[4]{81} \cdot \sqrt[4]{2}}{\sqrt[4]{16} \cdot \sqrt[4]{5}} = \frac{3\sqrt[4]{2}}{2\sqrt[4]{5}}$

Recombine the fraction under one radical sign $\quad = \frac{3}{2}\sqrt[4]{\frac{2}{5}}$

c) Here we are looking for the product of the highest perfect cube root and another number.

Often it is not very easy to identify the perfect root in the expression under the radical sign.

In this case, we can factor the number under the radical sign completely by using a factor tree.



We see that $135 = 3 \cdot 3 \cdot 3 \cdot 5 = 3^3 \cdot 5$

Therefore $\sqrt[3]{135} = \sqrt[3]{3^3 \cdot 5} = \sqrt[3]{3^3} \cdot \sqrt[3]{5} = 3\sqrt[3]{5}$

Here are some examples involving variables.

Example 5

Write the following expression in the simplest radical form.

a) $\sqrt{12x^3y^5}$

b) $\sqrt[4]{\frac{1250x^7}{405y^9}}$

Solution

Treat constants and each variable separately and write each expression as the products of a perfect power as indicated by the index of the radical and another number.

a)

$$\begin{aligned}
&\text{Rewrite as a product of radicals.} && \sqrt{12x^3y^5} = \sqrt{12} \cdot \sqrt{x^3} \cdot \sqrt{y^5} \\
&\text{Simplify each radical separately.} && \left(\sqrt{4 \cdot 3}\right) \cdot \left(\sqrt{x^2 \cdot x}\right) \cdot (y^4 \cdot y) = \left(2\sqrt{3}\right) \cdot (x\sqrt{x}) \cdot (y^2\sqrt{y}) \\
&\text{Combine all terms outside and inside the radical sign} && = 2xy^2\sqrt{3xy}
\end{aligned}$$

b)

$$\begin{aligned}
&\text{Rewrite as a quotient of radicals} && \sqrt[4]{\frac{1250x^7}{405y^9}} = \frac{\sqrt[4]{1250x^7}}{\sqrt[4]{405y^9}} \\
&\text{Simplify each radical separately} && = \frac{\sqrt[4]{625 \cdot 2} \cdot \sqrt[4]{x^4 \cdot x^3}}{\sqrt[4]{81 \cdot 5} \cdot \sqrt[4]{y^4 \cdot y^4 \cdot y}} = \frac{5\sqrt[4]{2} \cdot x \cdot \sqrt[4]{x^3}}{3\sqrt[4]{5} \cdot y \cdot \sqrt[4]{y}} = \frac{5x\sqrt[4]{2x^3}}{3y^2\sqrt[4]{5y}} \\
&\text{Recombine fraction under one radical sign} && = \frac{5x}{3y^2}\sqrt[4]{\frac{2x^3}{5y}}
\end{aligned}$$

Add and Subtract Radical Expressions

When we add and subtract radical expressions, we can combine radical terms only when they have the same expression under the radical sign. This is a similar procedure to combining like terms in variable expressions. For example,

$4\sqrt{2} + 5\sqrt{2} = 9\sqrt{2}$ or $2\sqrt{3} - \sqrt{2} + 5\sqrt{3} + 10\sqrt{2} = 7\sqrt{3} + 9\sqrt{2}$

It is important to simplify all radicals to their simplest form in order to make sure that we are combining all possible like terms in the expression. For example, the expression $\sqrt{8} - 2\sqrt{50}$ looks like it cannot be simplified any more because it has no like terms. However, when we write each radical in its simplest form we have

$2\sqrt{2} - 10\sqrt{2}$

This can be combined to obtain

$-8\sqrt{2}$

Example 6

Simplify the following expressions as much as possible.

a) $4\sqrt{3}+2\sqrt{12}$

b) $10\sqrt{24}-\sqrt{28}$

Solution

a)

$$\text{Simplify } \sqrt{12} \text{ to its simplest form.} = 4\sqrt{3}+2\sqrt{4\cdot 3} = 4\sqrt{3}+2\cdot 2\sqrt{3} = 4\sqrt{3}+4\sqrt{3}$$
$$\text{Combine like terms.} = 8\sqrt{3}$$

b)

$$\text{Simplify } \sqrt{24} \text{ and } \sqrt{28} \text{ to their simplest form.} = 10\sqrt{6\cdot 4}-\sqrt{7\cdot 4} = 20\sqrt{6}-2\sqrt{7}$$

There are no like terms.

Example 7

Simplify the following expressions as much as possible.

a) $4\sqrt[3]{128}-3\sqrt[3]{250}$

b) $3\sqrt{x^3}-4x\sqrt{9x}$

Solution

a)

$$\text{Rewrite radicals in simplest terms.} = 4\sqrt[3]{2.64}-\sqrt[3]{2.125} = 16\sqrt[3]{2}-5\sqrt[3]{2}$$
$$\text{Combine like terms.} = 11\sqrt[3]{2}$$

b)

$$\text{Rewrite radicals in simplest terms.} = 3\sqrt{x^2\cdot x}-4x\sqrt{9x} = 3x\sqrt{x}-12x\sqrt{x}$$
$$\text{Combine like terms.} = -9x\sqrt{x}$$

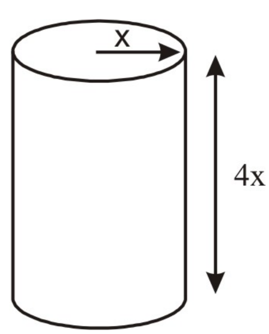
Solve Real-World Problems Using Radical Expressions

Radicals often arise in problems involving areas and volumes of geometrical figures.

Example 10

The volume of a soda can is 355 cm^3. The height of the can is four times the radius of the base. Find the radius of the base of the cylinder.

Solution



1. Make a sketch.
2. Let $x =$ the radius of the cylinder base
3. Write an equation.

The volume of a cylinder is given by

$$V = \pi r^2 \cdot h$$

4. Solve the equation.

$$\begin{aligned} 355 &= \pi x^2 (4x) \\ 355 &= 4\pi x^3 \\ x^3 &= \frac{355}{4\pi} \\ x &= \sqrt[3]{\frac{355}{4\pi}} = 3.046\ cm \end{aligned}$$

5. Check by substituting the result back into the formula.

$$V = \pi r^2 \cdot h = \pi (3.046)^2 \cdot (4 \cdot 3 \cdot 046) = 355\ cm^3$$

So the volume is 355 cm^3.

The answer checks out.

Review Questions

Evaluate each radical expression.

1. $\sqrt{169}$
2. $\sqrt[4]{81}$
3. $\sqrt[3]{-125}$
4. $\sqrt[5]{1024}$

Write each expression as a rational exponent.

5. $\sqrt[3]{14}$
6. $\sqrt[4]{zw}$
7. $\sqrt{a}$
8. $\sqrt[9]{y^3}$

Write the following expressions in simplest radical form.

9. $\sqrt{24}$
10. $\sqrt{300}$
11. $\sqrt[5]{96}$

12. $\sqrt{\frac{240}{567}}$
13. $\sqrt[3]{500}$
14. $\sqrt[6]{64x^8}$
15. $\sqrt[3]{48a^3b^7}$
16. $\sqrt[3]{\frac{16x^5}{135y^4}}$

Simplify the following expressions as much as possible.

17. $3\sqrt{8}-6\sqrt{32}$
18. $\sqrt{180}+6\sqrt{405}$
19. $\sqrt{6}-\sqrt{27}+2\sqrt{54}+3\sqrt{48}$
20. $\sqrt{8x^3}-4x\sqrt{98x}$
21. $\sqrt{48a}+\sqrt{27a}$
22. $\sqrt[3]{4x^3}+x\sqrt[3]{256}$
23. The volume of a spherical balloon is 950 cm-cubed. Find the radius of the balloon. (Volume of a sphere $=\frac{4}{3}\pi R^3$).

Review Answers

1. 13
2. not a real solution
3. -5
4. 4
5. $14^{\frac{1}{3}}$
6. $z^{\frac{1}{4}}w^{\frac{1}{4}}$
7. $a^{\frac{1}{2}}$
8. $y^{\frac{1}{3}}$
9. $2\sqrt{6}$
10. $10\sqrt{3}$
11. $2\sqrt[5]{3}$
12. $\frac{4}{9}\sqrt{\frac{15}{7}}$
13. $5\sqrt[3]{4}$
14. $2x.\sqrt[6]{x^2}$
15. $2ab^2\sqrt[3]{\sqrt{6}b}$
16. $\frac{2x}{3y}\sqrt{\frac{x^2}{5y}}$
17. $-18\sqrt{2}$
18. $15\sqrt{5}$
19. $7\sqrt{6}+9\sqrt{3}$
20. $-26x\sqrt{2x}$
21. $7\sqrt{3a}$
22. $5x\sqrt[3]{4}$
23. $R=6.1\ cm$

2.3 Radical Expressions II

Learning objectives

- Multiply radical expressions.
- Rationalize the denominator.

Multiply Radical Expressions.

When we multiply radical expressions, we use the "raising a product to a power" rule $\sqrt[m]{x \cdot y} = \sqrt[m]{x} \cdot \sqrt[m]{y}$.

In this case we apply this rule in reverse. For example

$$\sqrt{6} \cdot \sqrt{8} = \sqrt{6 \cdot 8} = \sqrt{48}$$

Make sure that the answer is written in simplest radical form

$$\sqrt{48} = \sqrt{16 \cdot 3} = 4\sqrt{3}$$

We will also make use of the fact that

$$\sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a.$$

When we multiply expressions that have numbers on both the outside and inside the radical sign, we treat the numbers outside the radical sign and the numbers inside the radical sign separately.

For example

$$a\sqrt{b} \cdot c\sqrt{d} = ac\sqrt{bd}.$$

Example 1

Multiply the following expressions.

a) $\sqrt{2}\left(\sqrt{3} + \sqrt{5}\right)$

b) $\sqrt{5}\left(5\sqrt{3} + 2\sqrt{5}\right)$

c) $2\sqrt{x}(3\sqrt{y} + \sqrt{x})$

Solution

In each case we use distribution to eliminate the parenthesis.

a)

$$\text{Distribute } \sqrt{2} \text{ inside the parenthesis.} \quad \sqrt{2}\left(\sqrt{3}+\sqrt{5}\right) = \sqrt{2}\cdot\sqrt{3}+\sqrt{2}\cdot\sqrt{5}$$

$$\text{Use the "raising a product to a power" rule.} \quad = \sqrt{2}\cdot\sqrt{3}+\sqrt{2}\cdot\sqrt{5}$$

$$\text{Simplify.} \quad = \sqrt{6}+\sqrt{10}$$

b)

$$\text{Distribute } \sqrt{5} \text{ inside the parenthesis.} \quad = 5\sqrt{5}\cdot\sqrt{3}-2\sqrt{5}\cdot\sqrt{5}$$

$$\text{Use the "raising a product to a power" rule.} \quad 5\sqrt{5\cdot 3}-2\sqrt{5\cdot 5} = 5\sqrt{15}-2\sqrt{25}$$

$$\text{Simplify.} \quad 5\sqrt{15}-2\cdot 5 = 5\sqrt{15}-10$$

c)

$$\text{Distribute } 2\sqrt{x} \text{ inside the parenthesis.} = (2\cdot 3)\left(\sqrt{x}\cdot\sqrt{y}\right)-2\cdot\left(\sqrt{x}\cdot\sqrt{x}\right)$$

$$\text{Multiply.} = 6\sqrt{xy}-2\sqrt{x^2}$$

$$\text{Simplify.} = 6\sqrt{xy}-2x$$

Example 2

Multiply the following expressions.

a) $\left(2+\sqrt{5}\right)\left(2-\sqrt{6}\right)$

b) $\left(2\sqrt{x}-1\right)\left(5-\sqrt{x}\right)$

Solution

In each case we use distribution to eliminate the parenthesis.

a)

$$\text{Distribute the parenthesis.} \quad \left(2+\sqrt{5}\right)\left(2-\sqrt{6}\right) = (2.2)-\left(2.\sqrt{6}\right)+\left(2.\sqrt{5}\right)-\left(\sqrt{5}.\sqrt{6}-\sqrt{30}\right)$$

$$\text{Simplify.} \quad 4-2\sqrt{6}+2\sqrt{5}-30$$

b)

$$\text{Distribute.} \quad \left(2\sqrt{x}-1\right)\left(5-\sqrt{x}\right) = 10\sqrt{x}-2x-5+\sqrt{x}$$

$$\text{Simplify.} \quad 11\sqrt{x}-2x-5$$

Rationalize the Denominator

Often when we work with radicals, we end up with a radical expression in the denominator of a fraction. We can simplify such expressions even further by eliminating the radical expression from the denominator of the expression. This process is called **rationalizing the denominator**.

There are two cases we will examine.

Case 1 There is a single radical expression in the denominator $\frac{2}{\sqrt{3}}$.

In this case, we multiply the numerator and denominator by a radical expression that makes the expression inside the radical into a perfect power. In the example above, we multiply by the $\sqrt{3}$.

$$\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

Next, let's examine $\frac{7}{\sqrt[3]{5}}$.

In this case, we need to make the number inside the cube root a perfect cube. We need to multiply the numerator and the denominator by $\sqrt[3]{5^2}$.

$$\frac{7}{\sqrt[3]{5}} \cdot \frac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}} = \frac{7^3\sqrt{25}}{\sqrt[3]{5^3}} = \frac{7^3\sqrt{25}}{5}$$

Case 2 The expression in the denominator is a radical expression that contains more than one term.

Consider the expression $\frac{2}{2+\sqrt{3}}$

In order to eliminate the radical from the denominator, we multiply it by $\left(2-\sqrt{3}\right)$. This is a good choice because the product $\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right)$ is a product of a sum and a difference which multiplies as follows.

$$\left(2+\sqrt{3}\right)\left(2-\sqrt{3}\right) = 2^2 - \left(\sqrt{3}\right)^2 = 4-3 = 1$$

We multiply the numerator and denominator by $\left(2-\sqrt{3}\right)$ and get

$$\frac{2}{2+\sqrt{3}} \cdot \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{2\left(2-\sqrt{3}\right)}{4-3} = \frac{4-2\sqrt{3}}{1}$$

Now consider the expression $\frac{\sqrt{x}-1}{\sqrt{x}-2\sqrt{y}}$.

In order to eliminate the radical expressions in the denominator, we must multiply by $\sqrt{x}+2\sqrt{y}$.

We obtain

$$\frac{\sqrt{x}-1}{\sqrt{x}-2\sqrt{y}} \cdot \frac{\sqrt{x}+2\sqrt{y}}{\sqrt{x}+2\sqrt{y}} = \frac{(\sqrt{x}-1)(\sqrt{x}+2\sqrt{y})}{(\sqrt{x}-2\sqrt{y})(\sqrt{x}+2\sqrt{y})}$$
$$= \frac{x+\sqrt{x}-2\sqrt{xy}-2\sqrt{y}}{x-4y}$$

Review Questions

Multiply the following expressions.

1. $\sqrt{6}\left(\sqrt{10}+\sqrt{8}\right)$

2. $\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{a}+\sqrt{b}\right)$
3. $\left(2\sqrt{x}+5\right)\left(2\sqrt{x}+5\right)$

Multiply the following expressions.

4. $\frac{7}{\sqrt{15}}$
5. $\frac{9}{\sqrt{10}}$
6. $\frac{2x}{\sqrt{5x}}$
7. $\frac{\sqrt{5}}{\sqrt{3y}}$
8. $\frac{12}{2-\sqrt{5}}$
9. $\frac{6-\sqrt{3}}{4-\sqrt{3}}$
10. $\frac{x}{\sqrt{2}+\sqrt{x}}$
11. $\frac{5y}{2\sqrt{y}-5}$

Review Answers

1. $2\sqrt{15}+4\sqrt{3}$
2. $a-b$
3. $4x+20\sqrt{x}+25$
4. $\frac{7\sqrt{5}}{5}$
5. $\frac{9\sqrt{10}}{10}$
6. $\frac{2\sqrt{5x}}{5}$
7. $\frac{\sqrt{15y}}{3y}$
8. $-24-12\sqrt{5}$
9. $\frac{27+10\sqrt{3}}{13}$
10. $\frac{\sqrt{2x}-x}{2-x}$
11. $\frac{10y\sqrt{y}+25y}{4y-25}$

2.4 Radical Equations

Learning Objectives

- Solve a radical equation.
- Solve radical equations with radicals on both sides.
- Identify extraneous solutions.
- Solve real-world problems using square root functions.

Introduction

When the variable in an equation appears inside a radical sign, the equation is called a **radical equation.** The first steps in solving a radical equation are to perform operations that will eliminate the radical and change the equation into a polynomial equation. A common method for solving radical equations is to isolate the most complicated radical on one side of the equation and raise both sides of the equation to the power that will eliminate the radical sign. If there are any radicals left in the equation after simplifying, we can repeat this procedure until all radical signs are gone. Once the equation is changed into a polynomial equation, we can solve it with the methods we already know.

We must be careful when we use this method, because whenever we raise an equation to a power, we could introduce false solutions that are not in fact solutions to the original problem. These are called **extraneous solutions.** In order to make sure we get the correct solutions, we must always check all solutions in the original radical equation.

Solve a Radical Equation

Let's consider a few simple examples of radical equations where only one radical appears in the equation.

Example 1

Find the real solutions of the equation $\sqrt{2x-1}=5$.

Solution

Since the radical expression is already isolated, we square both sides of the equation in order to eliminate the radical sign

$$\left(\sqrt{2x-1}\right)^2=5^2$$

$$\begin{aligned}\text{Remember that } \left(\sqrt{a}\right)^2=a \text{ so the equation simplifies to} \quad & 2x-1=25\\ \text{Add one to both sides.} \quad & 2x=26\\ \text{Divide both sides by 2.} \quad & amp;x=13\end{aligned}$$

Finally, we need to plug the solution in the original equation to see if it is a valid solution.

$$\sqrt{2x-1}=\sqrt{2(13)-1}=\sqrt{26-1}=\sqrt{25}=5$$

The answer checks out.

Example 2

Find the real solutions of $\sqrt[3]{3-7x}-3=0$.

Solution

We isolate the radical on one side of the equation.

$$\sqrt[3]{3-7x}=3$$

Raise each side of the equation to the third power.

$$\left(\sqrt[3]{3-7x}\right)^3=3^3$$

Simplify.

$$3-7x=27$$

Subtract 3 from each side.

$$-7x=24$$

Divide both sides by -7.

$$x=-\frac{24}{7}$$

Check

$$\sqrt[3]{3-7x}-3=\sqrt[3]{3-7\left(-\frac{24}{7}\right)}-3=\sqrt[3]{3+24}-3=\sqrt[3]{27}-3=3-3=0.$$

The answer checks out.

Example 3

Find the real solutions of $\sqrt{10-x^2}-x=2$.

Solution

We isolate the radical on one side of the equation.	$\sqrt{10-x^2}=2+x$
Square each side of the equation.	$\left(\sqrt{10-x^2}\right)=(2+x)^2$
Simplify.	$10-x^2=4+4x+x^2$
Move all terms to one side of the equation.	$0=2x^2+4x-6$
Solve using the quadratic formula.	$amp;x=\frac{-4\pm\sqrt{4^2-4(2)(-6)}}{6}$
Simplify.	$\frac{-4\pm\sqrt{64}}{4}$
Rewrite $\sqrt{64}$ in simplest form.	$amp;x=\frac{-4\pm 8}{4}$
Reduce all terms by a factor of 2.	$x=1$ or $x=-3$

Check

$$\sqrt{10-1^2}-1=\sqrt{9}-1=3-1=2$$

The answer checks out.

$$\sqrt{10-(-3)^2}-(-3)=\sqrt{1}+3=1+3=4\neq 2$$

This solution does not check out.

The equation has only one solution, $x=1$. The solution $x=-3$ is called an **extraneous** solution.

Solve Radical Equations with Radicals on Both Sides

Often equations have more than one radical expression. The strategy in this case is to isolate the most complicated radical expression and raise the equation to the appropriate power. We then repeat the process until all radical signs are eliminated.

Example 4

Find the real roots of the equation $\sqrt{2x+1}-\sqrt{x-3}=2$.

Solution

Isolate one of the radical expressions

$$\sqrt{2x+1}=2+\sqrt{x-3}$$

Square both sides

$$\left(\sqrt{2x+1}\right)^2=\left(2+\sqrt{x-3}\right)^2$$

Eliminate parentheses

$$2x+1 = 4+4\sqrt{x-3}+x-3$$

Simplify.

$$x = 4\sqrt{x-3}$$

Square both sides of the equation.

$$x^2 = \left(4\sqrt{x-3}\right)^2$$

Eliminate parentheses.

$$x^2 = 16(x-3)$$

Simplify.

$$x^2 = 16x-48$$

Move all terms to one side of the equation.

$$x^2-16x+48 = 0$$

Factor.

$$(x-12)(x-4) = 0$$

Solve.

$$x = 12 \text{ or } x = 4$$

Check

$$\sqrt{2(12)+1}-\sqrt{12-3} = \sqrt{25}-\sqrt{9} = 5-3 = 2$$

The solution checks out.

$$\sqrt{2(4)+1}-\sqrt{4-3} = \sqrt{9}-\sqrt{1} = 3-1 = 2$$

The solution checks out.

The equation has two solutions: $x = 12$ and $x = 4$.

Identify Extraneous Solutions to Radical Equations

We saw in Example 3 that some of the solutions that we find by solving radical equations do not check out when we substitute (or "plug in") those solutions back into the original radical equation. These are called **extraneous solutions**. It is very important to check the answers we obtain by plugging them back into the original equation. In this way, we can distinguish between the real and the extraneous solutions of an equation.

Example 5

Find the real roots of the equation $\sqrt{x-3}-\sqrt{x}=1$.

Solution

Isolate one of the radical expressions.

$$\sqrt{x-3}=\sqrt{x}+1$$

Square both sides.

$$\left(\sqrt{x-3}\right)^2=\left(\sqrt{x}+1\right)^2$$

Remove parenthesis.

$$x-3=\left(\sqrt{x}\right)^2+2\sqrt{x}+1$$

Simplify.

$$x-3=x+2\sqrt{x}+1$$

Now isolate the remaining radical.

$$-4=2\sqrt{x}$$

Divide all terms by 2.

$$-2=\sqrt{x}$$

Square both sides.

$$x=4$$

Check

$$\sqrt{4-3}-\sqrt{4}=\sqrt{1}-2=1-2=-1$$

The solution does not check out.

The equation has no real solutions. Therefore, $x=4$ is an extraneous solution.

Solve Real-World Problems using Radical Equations

Radical equations often appear in problems involving areas and volumes of objects.

Example 6

The area of Anita's square vegetable garden is 21 square-feet larger that Fred's square vegetable garden. Anita and Fred decide to pool their money together and buy the same kind of fencing for their gardens. If they need 84 feet of fencing, what is the size of their gardens?

Solution

1. **Make a sketch**
2. **Define variables**

Let Fred's area be x

Anita's area $x+21$

Therefore,

Side length of Fred's garden is $\sqrt{x}$

Side length of Anita's garden is $\sqrt{x+21}$

3. Find an equation

The amount of fencing is equal to the combined perimeters of the two squares.

$$4\sqrt{x}+4\sqrt{x+21}=84$$

4. Solve the equation

Divide all terms by 4.

$$\sqrt{x}+\sqrt{x+21}=21$$

Isolate one of the radical expressions.

$$\sqrt{x+21}=21-\sqrt{x}$$

Square both sides.

$$\left(\sqrt{x+21}\right)^2=\left(21-\sqrt{x}\right)^2$$

Eliminate parentheses.

$$x+21=441-42\sqrt{x}+x$$

Isolate the radical expression.

$$42\sqrt{x}=420$$

Divide both sides by 42.

$$\sqrt{x}=10$$

Square both sides.

$$x=100\ ft^2$$

5. Check

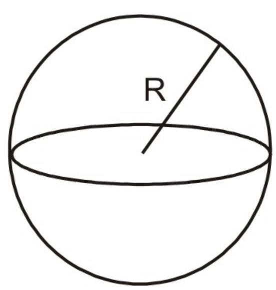
$$4\sqrt{100}+4\sqrt{100+21}=40+44=84$$

The solution checks out.

Fred's garden is $10\ ft \times 10\ ft = 100\ ft^2$ and Anita's garden is $11\ ft \times 11\ ft = 121\ ft^2$.

Example 7

A sphere has a volume of $456\ cm^3$. *If the radius of the sphere is increased by 2 cm, what is the new volume of the sphere?*



Solution

1. **Make a sketch.** Let's draw a sphere.
2. **Define variables.** Let $R =$ the radius of the sphere.
3. **Find an equation.**

The volume of a sphere is given by the formula:

$$V = \frac{4}{3}\pi r^3$$

4. Solve the equation.

Plug in the value of the volume.

$$456 = \frac{4}{3}\pi r^3$$

Multiply by 3.

$$1368 = 4\pi r^3$$

Divide by 4π.

$$108.92 = r^3$$

Take the cube root of each side.

$$r = \sqrt[3]{108.92} \Rightarrow r = 4.776\ cm$$

The new radius is 2 centimeters more.

$$r = 6.776\ cm$$

The new volume is:

$$V = \frac{4}{3}\pi(6.776)^3 = 1302.5\ cm^3$$

5. Check

Let's substitute in the values of the radius into the volume formula.

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(4.776)^3 = 456\ cm^3.$$

The solution checks out.

Example 8

The kinetic energy of an object of mass m and velocity v is given by the formula $KE = \frac{1}{2}mv^2$. *A baseball has a mass of 145 kg and its kinetic energy is measured to be 654 Joules* $(1\ Joule = 1\ kg \cdot m^2/s^2)$ *when it hits the catcher's glove. What is the velocity of the ball when it hits the catcher's glove?*

Solution

1. Start with the formula. $KE = \frac{1}{2}\,mv^2$
2. Plug in the values for the mass and the kinetic energy. $654\frac{kg \cdot m^2}{s^2} = \frac{1}{2}(145\ kg)v^2$
3. Multiply both sides by 2. $1308\frac{kg \cdot m^2}{s^2} = (145\ kg)v^2$
4. Divide both sides by 145 kg. $9.02\frac{m^2}{s^2} = v^2$
5. Take the square root of both sides. $v = \sqrt{9.02}\sqrt{\frac{m^2}{s^2}} = 3.003\ m/s$
6. **Check** Plug the values for the mass and the velocity into the energy formula.

$$KE = \frac{1}{2}\,mv^2 = \frac{1}{2}(145\ kg)(3.003\ m/s)^2 = 654\ kg \cdot m^2/s^2$$

Review Questions

Find the solution to each of the following radical equations. Identify extraneous solutions.

1. $\sqrt{x+2} - 2 = 0$
2. $\sqrt{3x-1} = 5$
3. $2\sqrt{4-3x} + 3 = 0$
4. $\sqrt[3]{x-3} = 1$
5. $\sqrt[4]{x^2-9} = 2$
6. $\sqrt[3]{-2-5x} + 3 = 0$
7. $\sqrt{x} = x - 6$
8. $\sqrt{x^2-5x} - 6 = 0$
9. $\sqrt{(x+1)(x-3)} = x$
10. $\sqrt{x+6} = x + 4$
11. $\sqrt{x} = \sqrt{x-9} + 1$
12. $\sqrt{3x+4} = -6$
13. $\sqrt{10-5x} + \sqrt{1-x} = 7$
14. $\sqrt{2x-2} - 2\sqrt{x} + 2 = 0$
15. $\sqrt{2x+5} - 3\sqrt{2x-3} = \sqrt{2-x}$
16. $3\sqrt{x} - 9 = \sqrt{2x-14}$
17. The area of a triangle is 24 in^2 and the height of the triangle is twice as long and the base. What are the base and the height of the triangle?
18. The area of a circular disk is 124 in^2. What is the circumference of the disk? (Area $= \pi r^2$, Circumference $= 2\pi r$).
19. The volume of a cylinder is 245 cm^3 and the height of the cylinder is one third of the diameter of the base of the cylinder. The diameter of the cylinder is kept the same, but the height of the cylinder is increased by two centimeters. What is the volume of the new cylinder? (Volume $= \pi r^2 \cdot h$)
20. The height of a golf ball as it travels through the air is given by the equation $h = -16t^2 + 256$. Find the time when the ball is at a height of 120 feet.

Review Answers

1. $x = 2$
2. $x = \frac{26}{3}$
3. No real solution, extraneous solution $x = \frac{7}{12}$
4. $x = 4$
5. $x = 5$ or $x = -5$

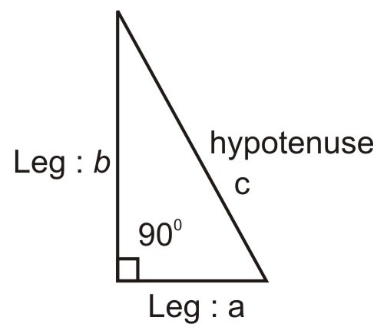
6. $x = 5$
7. $x = 9$, extraneous solution $x = 4$
8. $x = 9$ or $x = -4$
9. No real solution, extraneous solution $x = -\frac{3}{2}$
10. $x = -2$, extraneous solution $x = -5$
11. $x = 25$
12. No real solution, extraneous solution $x = \frac{32}{3}$
13. $x = -3$, extraneous solution $x = -\frac{117}{4}$
14. $x = 9, x = 1$
15. $x = 2, x = \frac{62}{33}$
16. $x = 25$, extraneous solution $x = \frac{361}{49}$
17. Base $= 4.9$ *in*, Height $= 9.8$ *in*
18. Circumference $= 39.46$ *in*
19. Volume $= 394.94\ cm^3$
20. Time $= 2.9$ *seconds*

2.5 The Pythagorean Theorem and Its Converse

Learning Objectives

- Use the Pythagorean Theorem.
- Use the converse of the Pythagorean Theorem.
- Solve real-world problems using the Pythagorean Theorem and its converse.

Introduction



The **Pythagorean Theorem** is a statement of how the lengths of the sides of a right triangle are related to each other. A right triangle is one that contains a 90 degree angle. The side of the triangle opposite the 90 degree angle is called the **hypotenuse** and the sides of the triangle adjacent to the 90 degree angle are called the **legs**.

If we let a and b represent the legs of the right triangle and c represent the hypotenuse, then the Pythagorean Theorem can be stated as:

In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs.

That is,

$$(\text{leg}_1)^2 + (\text{leg}_2)^2 = (\text{hypotenuse})^2$$

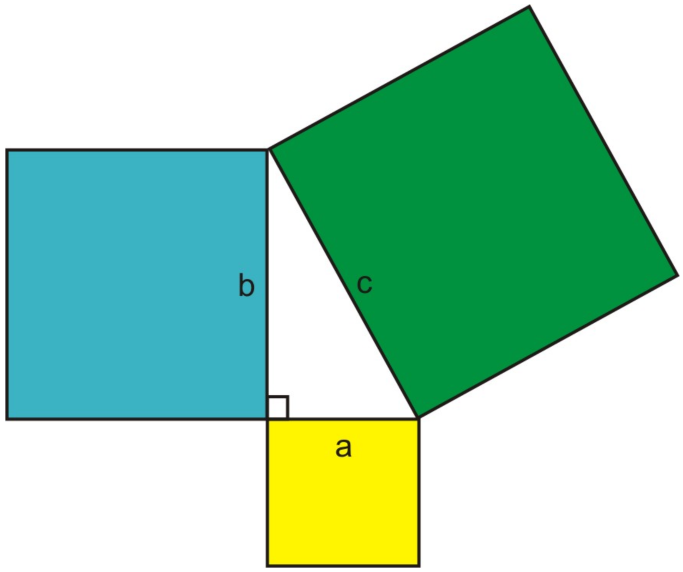
Or using the labels given in the triangle to the right

$$a^2 + b^2 = c^2$$

This theorem is very useful because if we know the lengths of the legs of a right triangle, we can find the length of the hypotenuse. **Conversely**, if we know the length of the hypotenuse and the length of a leg, we can calculate the length of the missing leg of the triangle. When you use the Pythagorean Theorem, it does not matter which leg you call a and which leg you call b, but the hypotenuse is always called c.

Although nowadays we use the Pythagorean Theorem as a statement about the relationship between distances and lengths, originally the theorem made a statement about areas. If we build squares on each side of a right triangle, the

Pythagorean Theorem says that the area of the square whose side is the hypotenuse is equal to the sum of the areas of the squares formed by the legs of the triangle.



Use the Pythagorean Theorem and Its Converse

The Pythagorean Theorem can be used to verify that a triangle is a right triangle. If you can show that the three sides of a triangle make the equation $(\text{leg}_1)^2 + (\text{leg}_2)^2 = (\text{hypotenuse})^2$ true, then you know that the triangle is a right triangle. This is called the **Converse of the Pythagorean Theorem**.

Note: When you use the Converse of the Pythagorean Theorem, you must make sure that you substitute the correct values for the legs and the hypotenuse. One way to check is that the hypotenuse must be the longest side. The other two sides are the legs and the order in which you use them is not important.

Example 1

Determine if a triangle with sides 5, 12 and 13 is a right triangle.

Solution

The triangle is right if its sides satisfy the Pythagorean Theorem.

First of all, the longest side would have to be the hypotenuse so we designate $c = 13$.

We then designate the shorter sides as $a = 5$ and $b = 12$.

We plug these values into the Pythagorean Theorem.

$$a^2 + b^2 = c^2 \Rightarrow 5^2 + 12^2 = c^2$$
$$25 + 144 = 169 = c^2 \Rightarrow 169 = 169$$

The sides of the triangle satisfy the Pythagorean Theorem, thus the triangle is a right triangle.

Example 2

Determine if a triangle with sides $\sqrt{10}$, $\sqrt{15}$ *and 5 is a right triangle.*

Solution

We designate the hypotenuse $c = 5$ because this is the longest side.

We designate the shorter sides as $a = \sqrt{10}$ and $b = \sqrt{15}$.

We plug these values into the Pythagorean Theorem.

$$a^2 + b^2 = c^2 \Rightarrow \left(\sqrt{10}\right)^2 + \left(\sqrt{15}\right)^2 = c^2$$
$$10 + 15 = 25 = (5)^2$$

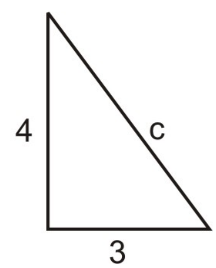
The sides of the triangle satisfy the Pythagorean Theorem, thus the triangle is a right triangle.

Pythagorean Theorem can also be used to find the missing hypotenuse of a right triangle if we know the legs of the triangle.

Example 3

In a right triangle one leg has length 4 and the other has length 3. Find the length of the hypotenuse.

Solution



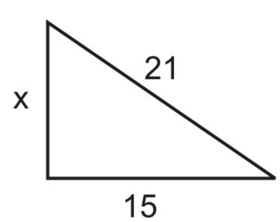
Start with the Pythagorean Theorem.	$amp; a^2 + b^2 = c^2$
Plug in the known values of the legs.	$3^2 + 4^2 = c^2$
Simplify.	$9 + 16 = c^2$
	$25 = c^2$
Take the square root of both sides.	$amp; c = 5$

Use the Pythagorean Theorem with Variables

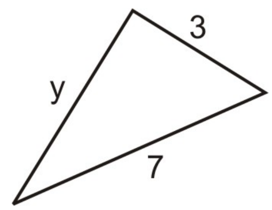
Example 4

Determine the values of the missing sides. You may assume that each triangle is a right triangle.

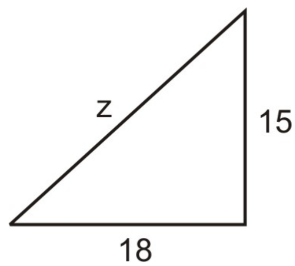
a)



b)



c)



Solution

Apply the Pythagorean Theorem.

a)

$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
x^2 + 15^2 &= 21^2 \\
x^2 + 225 &= 441 \\
x^2 &= 216 \Rightarrow \\
x &= \sqrt{216} = 6\sqrt{6}
\end{aligned}
$$

b)

$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
y^2 + 3^2 &= 7^2 \\
y^2 + 9 &= 49 \\
y^2 &= 40 \Rightarrow \\
y &= \sqrt{40} = 2\sqrt{10}
\end{aligned}
$$

c)

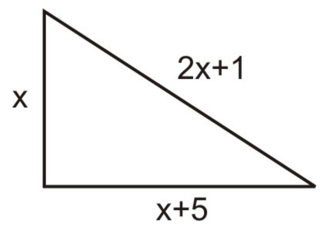
$$
\begin{aligned}
a^2 + b^2 &= c^2 \\
18^2 + 15^2 &= z^2 \\
324 + 225 &= z^2 \\
z^2 &= 216 \Rightarrow \\
x &= \sqrt{549} = 3\sqrt{61}
\end{aligned}
$$

Example 5

One leg of a right triangle is 5 more than the other leg. The hypotenuse is one more than twice the size of the short leg. Find the dimensions of the triangle.

Solution

Let $x =$ length of the short leg.



Then, $x+5 =$ length of the long leg

And, $2x+1 =$ length of the hypotenuse.

The sides of the triangle must satisfy the Pythagorean Theorem,

$$\begin{aligned}
&\text{Therefore} && amp; x^2+(x+5)^2 = (2x+1) \\
&\text{Eliminate the parenthesis.} && amp; x^2+x^2+10x+25 = 4x^2+4x+1 \\
&\text{Move all terms to the right hand side of the equation.} && 0 = 2x^2-6x-24 \\
&\text{Divide all terms by 2.} && 0 = x^2-3x-12 \\
&\text{Solve using the quadratic formula.} && x = \frac{3 \pm \sqrt{9+48}}{2} = \frac{3 \pm \sqrt{57}}{2} \\
& && x \approx 5.27 \text{ or } x \approx -2.27
\end{aligned}$$

Answer

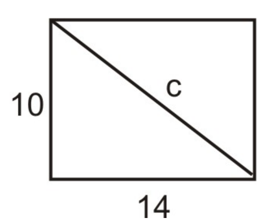
We can discard the negative solution since it does not make sense in the geometric context of this problem. Hence, we use $x = 5.27$ and we get short - leg $= 5.27$, long - leg $= 10.27$ and hypotenuse $= 11.54$.

Solve Real-World Problems Using the Pythagorean Theorem and Its Converse

The Pythagorean Theorem and its converse have many applications for finding lengths and distances.

Example 6

Maria has a rectangular cookie sheet that measures 10 *inches* $\times$ 14 *inches. Find the length of the diagonal of the cookie sheet.*



Solution

1. **Draw a sketch.**

2. **Define variables.**

Let $c =$ length of the diagonal.

3. **Write a formula.** Use the Pythagorean Theorem

$$a^2 + b^2 = c^2$$

4. **Solve the equation.**

$$\begin{aligned}
10^2 + 14^2 &= c^2 \\
100 + 196 &= c^2 \\
c^2 &= 296 \Rightarrow c = \sqrt{296} \Rightarrow c = 4\sqrt{74} \text{ or } c \approx 17.2 \textit{ inches}
\end{aligned}$$

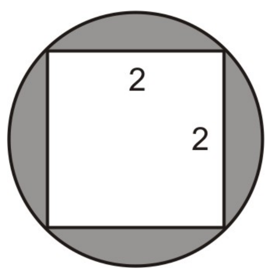
5. **Check**

$10^2 + 14^2 = 100 + 196$ and $c^2 = 17.2^2 \approx 296$.

The solution checks out.

Example 7

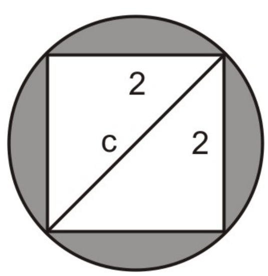
Find the area of the shaded region in the following diagram.



Solution:

1. **Diagram**

Draw the diagonal of the square on the figure.



Notice that the diagonal of the square is also the diameter of the circle.

2. **Define variables**

Let $c =$ diameter of the circle.

3. **Write the formula**

Use the Pythagorean Theorem: $a^2 + b^2 = c^2$

4. **Solve the equation:**

$$2^2 + 2^2 = c^2$$
$$4 + 4 = c^2$$
$$c^2 = 8 \Rightarrow c = \sqrt{8} \Rightarrow c = 2\sqrt{2}$$

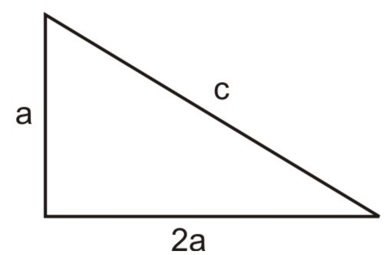
The diameter of the circle is $2\sqrt{2}$. Therefore, the radius is $r = \sqrt{2}$.

Area of a circle is $A = \pi r^2 = \pi\left(\sqrt{2}\right)^2 = 2\pi$.

Area of the shaded region is therefore $2\pi - 4 \approx 2.28$

Example 8

In a right triangle, one leg is twice as long as the other and the perimeter is 28. What are the measures of the sides of the triangle?



Solution

1. **Make a sketch.** Let's draw a right triangle.

2. **Define variables.**

Let: $a =$ length of the short leg

$2a =$ length of the long leg

$c =$ length of the hypotenuse

3. **Write formulas.**

The sides of the triangle are related in two different ways.

1. The perimeter is 28, $a + 2a + c = 28 \Rightarrow 3a + c = 28$

2. This a right triangle, so the measures of the sides must satisfy the Pythagorean Theorem.

$$a^2 + (2a)^2 = c^2 \Rightarrow a^2 + 4a^2 = c^2 \Rightarrow 5a^2 = c^2$$
$$\text{or}$$
$$c = a\sqrt{5} \approx 2.236a$$

4. **Solve the equation**

Use the value of c we just obtained to plug into the perimeter equation $3a + c = 28$.

$$3a + 2.236a = 28 \Rightarrow 5.236a = 28 \Rightarrow a = 5.35$$

The short leg is:s $a \approx 5.35$.

The long leg is: $2a \approx 10.70$.

The hypotenuse is: $c \approx 11.95$.

5. **Check** The legs of the triangle should satisfy the Pythagorean Theorem

$$a^2 + b^2 = 5.35^2 + 10.70^2 = 143.1, c^2 = 11.95^2 = 142.80$$

The results are approximately the same.

The perimeter of the triangle should be 28.

$$a + b + c = 5.35 + 10.70 + 11.95 = 28$$

The answer checks out.

Example 9

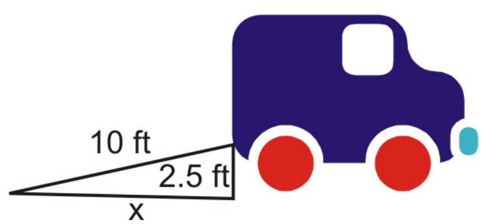
Mike is loading a moving van by walking up a ramp. The ramp is 10 feet long and the bed of the van is 2.5 feet above the ground. How far does the ramp extend past the back of the van?

Solution

1. **Make a sketch.**

2. **Define Variables.**

Let $x =$ how far the ramp extends past the back of the van.



3. **Write a formula.** Use the Pythagorean Theorem:

$$x^2 + 2.5^2 = 10^2$$

4. **Solve the equation.**

$$\begin{aligned} x^2 + 6.25 &= 100 \\ x^2 &= 93.5 \\ x &= \sqrt{93.5} \approx 9.7 \ ft \end{aligned}$$

5. **Check.** Plug the result in the Pythagorean Theorem.

$$9.7^2 + 2.5^2 = 94.09 + 6.25 = 100.36 \approx 100.$$

The ramp is 10 feet long.

The answer checks out.

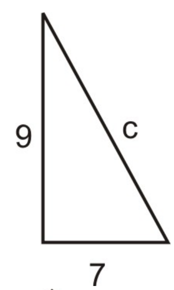
Review Questions

Verify that each triangle is a right triangle.

1. $a = 12, b = 9, c = 15$
2. $a = 6, b = 6, c = 6\sqrt{2}$
3. $a = 8, b = 8\sqrt{3}, c = 16$

Find the missing length of each right triangle.

4. $a = 12, b = 16, c = ?$
5. $a = ?, b = 20, c = 30$
6. $a = 4, b = ?, c = 11$



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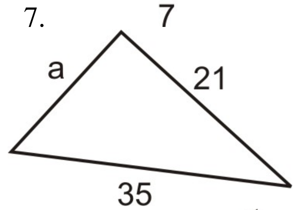

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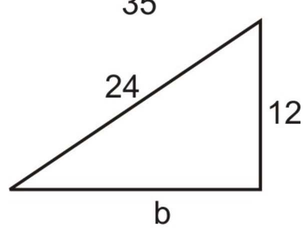
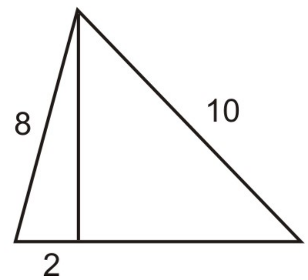


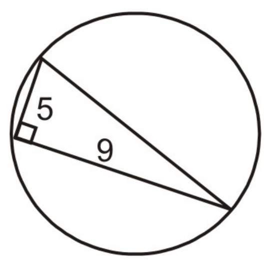
9.

10. One leg of a right triangle is 4 feet less than the hypotenuse. The other leg is 12 feet. Find the lengths of the three sides of the triangle.
11. One leg of a right triangle is 3 more than twice the length of the other. The hypotenuse is 3 times the length of the short leg. Find the lengths of the three legs of the triangle.
12. A regulation baseball diamond is a square with 90 feet between bases. How far is second base from home plate?
13. Emanuel has a cardboard box that measures $20\ cm \times 10\ cm \times 8\ cm$ (length × width × height). What is the length of the diagonal from a bottom corner to the opposite top corner?
14. Samuel places a ladder against his house. The base of the ladder is 6 feet from the house and the ladder is 10 feet long. How high above the ground does the ladder touch the wall of the house?

15. Find the area of the triangle if area of a triangle is defined as $A = \frac{1}{2}\text{base} \times \text{height}$.



16. Instead of walking along the two sides of a rectangular field, Mario decided to cut across the diagonal. He saves a distance that is half of the long side of the field. Find the length of the long side of the field given that the short side is 123 feet.
17. Marcus sails due north and Sandra sails due east from the same starting point. In two hours, Marcus' boat is 35 miles from the starting point and Sandra's boat is 28 miles from the starting point. How far are the boats from each other?
18. Determine the area of the circle.



Review Answers

1. $12^2 + 9^2 = 225$
   $15^2 = 225$
2. $6^2 + 6^2 = 72$
   $\left(6\sqrt{2}\right)^2 = 72$
3. $8^2 + \left(8\sqrt{3}\right)^2 = 256$
   $16^2 = 256$
4. $c = 20$
5. $a = 10\sqrt{5}$
6. $b = \sqrt{105}$
7. $c = \sqrt{130}$
8. $a = 28$
9. $b = 12\sqrt{3}$
10. 12, 16, 20
11. 3.62, 10.24, 10.86
12. 127.3 ft
13. 23.75 cm
14. 8 feet
15. 32.24
16. 164 feet
17. 44.82 miles
18. 83.25

2.6 Distance and Midpoint Formulas

Learning Objectives

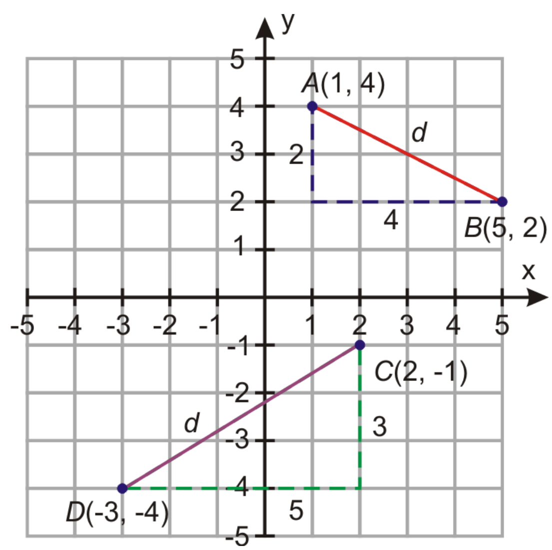
- Find the distance between two points in the coordinate plane.
- Find the missing coordinate of a point given the distance from another known point.
- Find the midpoint of a line segment.
- Solve real-world problems using distance and midpoint formulas.

Introduction

In the last section, we saw how to use the Pythagorean Theorem in order to find lengths. In this section, you will learn how to use the Pythagorean Theorem to find the distance between two coordinate points.

Example 1

Find distance between points $A = (1,4)$ *and* $B = (5,2)$.



Solution

Plot the two points on the coordinate plane. In order to get from point $A = (1,4)$ to point $B = (5,2)$, we need to move 4 units to the right and 2 units down.

To find the distance between A and B we find the value of d using the Pythagorean Theorem.

$$d^2 = 2^2 + 4^2 = 20$$
$$d = \sqrt{20} = 2\sqrt{5} = 4.47$$

Example 2: Find the distance between points $C = (2,1)$ and $D = (-3,-4)$.

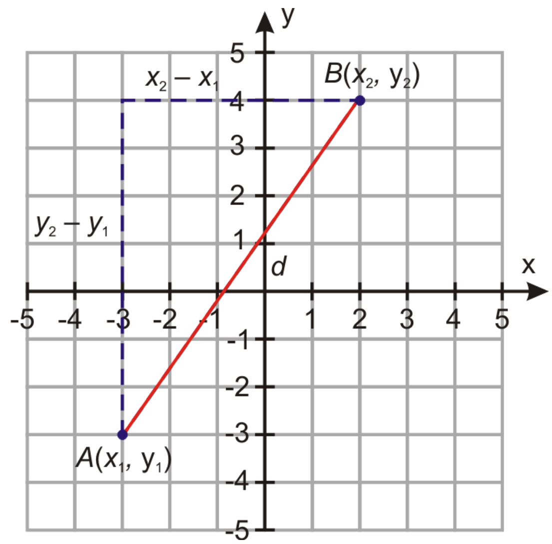
Solution: We plot the two points on the graph above.

In order to get from point C to point D, we need to move 3 units down and 5 units to the left.

We find the distance from C to D by finding the length of d with the Pythagorean Theorem.

$$d^2 = 3^2 + 5^2 = 34$$
$$d = \sqrt{34} = 5.83$$

The Distance Formula



This procedure can be generalized by using the Pythagorean Theorem to derive a formula for the distance between two points on the coordinate plane.

Let's find the distance between two general points $A = (x_1, y_1)$ and $B = (x_2, y_2)$.

Start by plotting the points on the coordinate plane.

In order to move from point A to point B in the coordinate plane, we move $x_2 - x_1$ units to the right and $y_2 - y_1$ units up. We can find the length d by using the Pythagorean Theorem.

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

This equation leads us to the Distance Formula.

Given two points (x_1, y_1) and (x_2, y_2) the distance between them is:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

We can use this formula to find the distance between two points on the coordinate plane. Notice that the distance is the same if you are going from point A to point B as if you are going from point B to point A. Thus, it does not matter which order you plug the points into the distance formula.

Find the Distance Between Two Points in the Coordinate Plane

Let's now apply the distance formula to the following examples.

Example 2

Find the distance between the following points.

a) (-3, 5) and (4, -2)

b) (12, 16) and (19, 21)

c) (11.5, 2.3) and (-4.2, -3.9)

Solution

Plug the values of the two points into the distance formula. Be sure to simplify if possible.

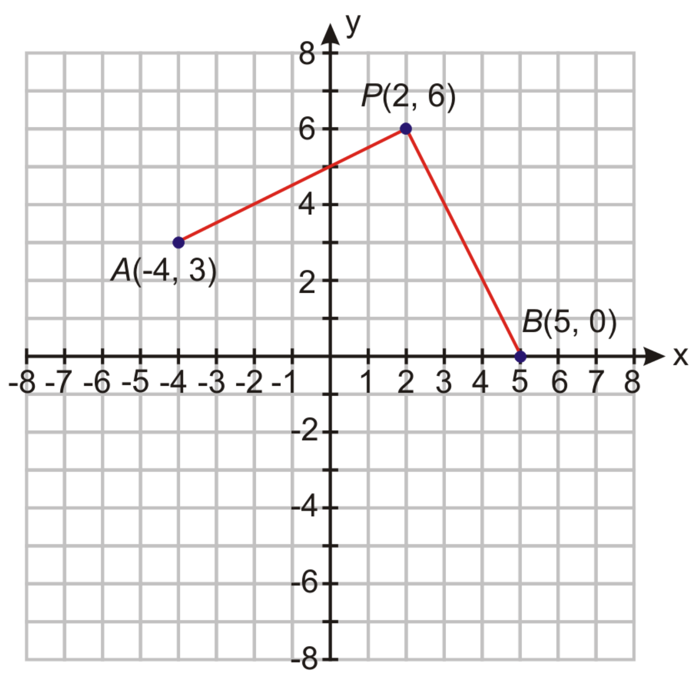
a) $d = \sqrt{(-3-4)^2+(5-(-2))^2} = \sqrt{(-7)^2+(7)^2} = \sqrt{49+49} = \sqrt{98} = 7\sqrt{2}$

b) $d = \sqrt{(12-19)^2+(16-21)^2} = \sqrt{(-7)^2+(-5)^2} = \sqrt{49+25} = \sqrt{74}$

c) $d = \sqrt{(11.5+4.2)^2+(2.3+3.9)^2} = \sqrt{(15.7)^2+(6.2)^2} = \sqrt{284.93} = 16.88$

Example 3

Show that point $P = (2,6)$ is equidistant for $A = (-4,3)$ and $B = (5,0)$.



Solution

To show that the point P is equidistant from points A and B, we need to show that the distance from P to A is equal to the distance from P to B.

Before we apply the distance formula, let's graph the three points on the coordinate plane to get a visual representation of the problem.

From the graph we see that to get from point P to point A, we move 6 units to the left and 3 units down. To move from point P to point B, we move 6 units down and 3 units to the left. From this information, we should expect P to be equidistant from A and B.

Now, let's apply the distance formula for find the lengths PA and PB.

$$PA = \sqrt{(2+4)^2+(6-3)^2} = \sqrt{(6)^2+(3)^2} = \sqrt{39+9} = \sqrt{45}$$
$$PA = \sqrt{(2-5)^2+(6-0)^2} = \sqrt{(-3)^2+(6)^2} = \sqrt{9+36} = \sqrt{45}$$

$PA = PB$, so P is equidistant from points A and B.

Find the Missing Coordinate of a Point Given Distance From Another Known Point

Example 4

Point $A = (6,-4)$ *and point* $B = (2,k)$*. What is the value of k such that the distance between the two points is 5?*

Solution

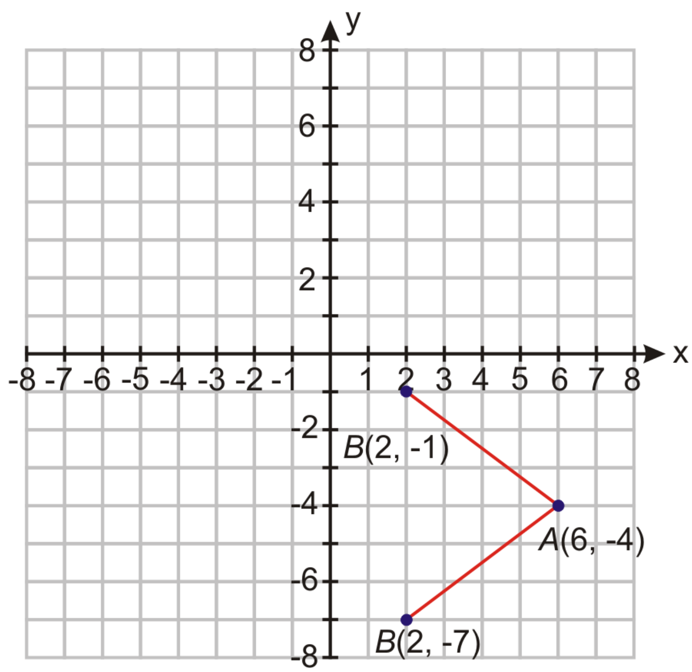
Let's use the distance formula.

$$d = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \Rightarrow 5 = \sqrt{(6-2)^2+(-4-k)^2}$$

Square both sides of the equation.	$5^2 = \left[\sqrt{(6-2)^2+(-4-k)^2}\right]^2$
Simplify.	$25 = 16+(-4-k)^2$
Eliminate the parentheses.	$0 = -9+k^2+8k+16$
Simplify.	$0 = k^2+8k+7$
Find k using the quadratic formula.	$amp;k = \frac{-8\pm\sqrt{64-28}}{2} = \frac{-8\pm\sqrt{36}}{2} = \frac{-8\pm6}{2}$

Answer $k = -7$ or $k = -1$.

Therefore, there are two possibilities for the value of k. Let's graph the points to get a visual representation of our results.



From the figure, we can see that both answers make sense because they are both equidistant from point A.

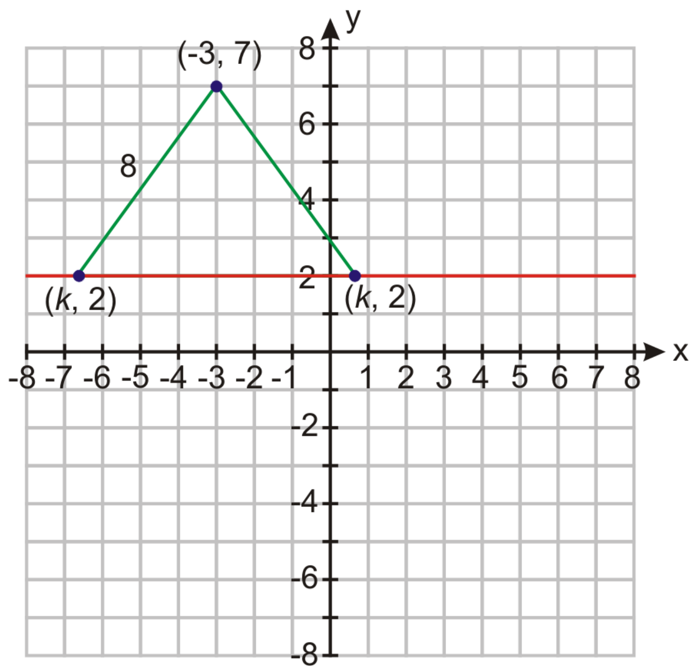
Example 5

Find all points on line $y = 2$ that are a distance of 8 units away from point (-3, 7).

Solution

Let's make a sketch of the given situation. Draw line segments from point (-3, 7) to the line $y = 2$. Let k be the missing value of x we are seeking. Let's use the distance formula.

$$8 = \sqrt{(-3-k)^2 + (7-2)^2}$$



Now let's solve using the distance formula.

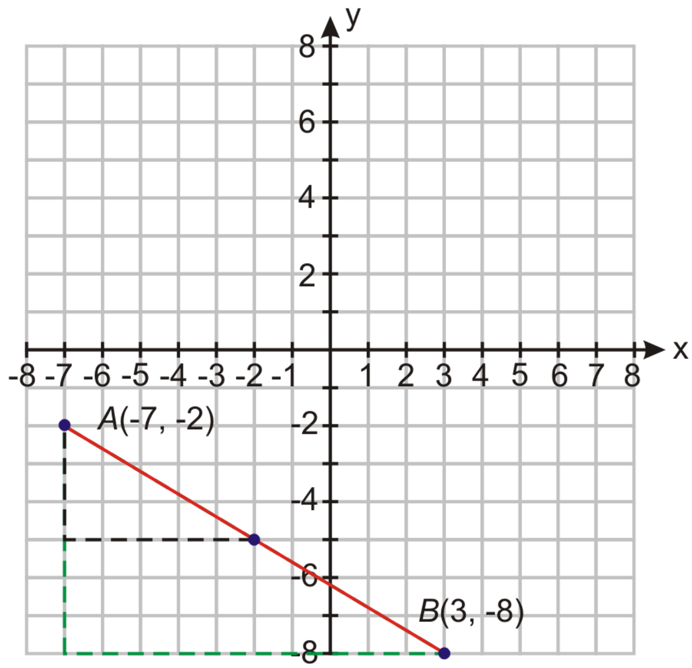
Square both sides of the equation	$64 = (-3-k)^2 + 25$
Therefore.	$0 = 9 + 6k + k^2 - 39$
Or	$0 = k^2 + 6k - 30$
Use the quadratic formula.	$amp;k = \frac{-6 \pm \sqrt{36+120}}{2} = \frac{-6 \pm \sqrt{156}}{2}$
Therefore.	$amp;k \approx 3.24 \text{ or } k \approx -9.24$

Answer The points are (-9.24, 2) and (3.24, 2)

Find the Midpoint of a Line Segment

Example 6

Find the coordinates of the point that is in the middle of the line segment connecting points $A = (-7, -2)$ and $B = (3, -8)$.



Solution

Let's start by graphing the two points.

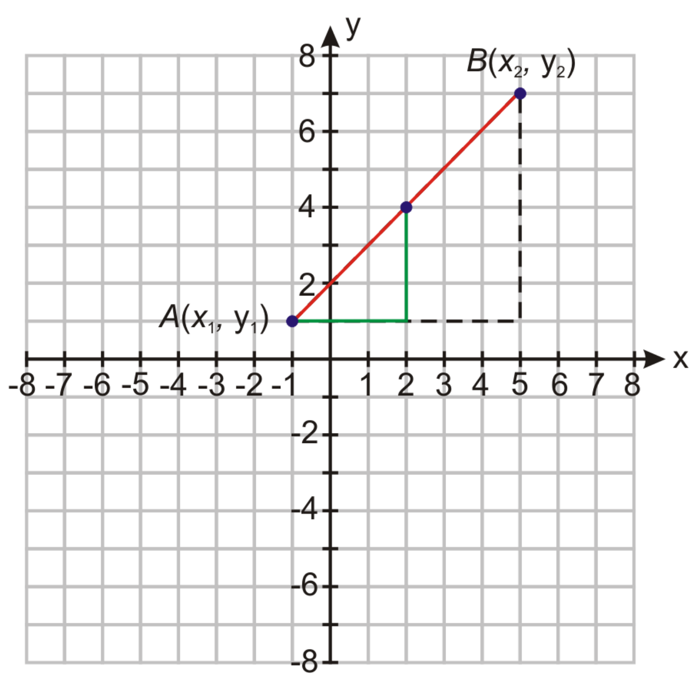
We see that to get from point A to point B we move 6 units down and 10 units to the right.

In order to get to the point that is half-way between the two points, it makes sense that we should move half the vertical and half the horizontal distance, that is 3 units down and 5 units to the right from point A.

The midpoint is $M = (-7+5, -2-3) = (-2, -5)$

The Midpoint Formula:

We now want to generalize this method in order to find a formula for the midpoint of a line segment.



Let's take two general points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ and mark them on the coordinate plane.

We see that to get from A to B, we move $x_2 - x_1$ units to the right and $y_2 - y_1$ up.

In order to get to the half-way point, we need to move

$\frac{x_2-x_1}{2}$ units to the right and $\frac{y_2-y_1}{2}$ up from point A.

Thus the midpoint, $M = \left(x_1 + \frac{x_2-x_1}{2}, y_1 + \frac{y_2-y_1}{2}\right)$.

This simplifies to: $M = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$. This is the **Midpoint Formula.**

It should hopefully make sense that the midpoint of a line is found by taking the average values of the x and $y-$values of the endpoints.

Midpoint Formula

The midpoint of the segment connecting points (x_1, y_1) and (x_2, y_2) has coordinates

$$M = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right).$$

Example 7

Find the midpoint between the following points.

a) (-10, 2) and (3, 5)

b) (3, 6) and (7, 6)

c) (4, -5) and (-4, 5)

Solution

Let's apply the Midpoint Formula.

$$M = \left(\frac{x_2+x_1}{2}, \frac{y_2+y_1}{2}\right)$$

a) The midpoint of (-10, 2) and (3, 5) is $\left(\frac{-10+3}{2}, \frac{2+5}{2}\right) = \left(\frac{-7}{2}, \frac{7}{2}\right) = (-3.5, 3.5)$.

b) The midpoint of (3, 6) and (7, 6) is $\left(\frac{3+7}{2}, \frac{6+6}{2}\right) = \left(\frac{10}{2}, \frac{12}{2}\right) = (5, 6)$.

c) The midpoint of (4, -5) and (-4, 5) is $\left(\frac{4-4}{2}, \frac{-5+5}{2}\right) = \left(\frac{0}{2}, \frac{0}{2}\right) = (0, 0)$.

Example 8

A line segment whose midpoint is (2, -6) has an endpoint of (9, -2). What is the other endpoint?

Solution

In this problem we know the midpoint and we are looking for the missing endpoint.

The midpoint is (2, -6).

One endpoint is $(x_1, y_1) = (9, -2)$

Let's call the missing point (x, y).

We know that the $x-$coordinate of the midpoint is 2, so

$$2 = \frac{9+x_2}{2} \Rightarrow 4 = 9 + x_2 \Rightarrow x_2 = -5$$

We know that the $y-$coordinate of the midpoint is -6, so

$$-6 = \frac{-2+y_2}{2} \Rightarrow -12 = -2+y_2 \Rightarrow y_2 = -10$$

Answer The missing endpoint is (-5, -10).

Solve Real-World Problems Using Distance and Midpoint Formulas

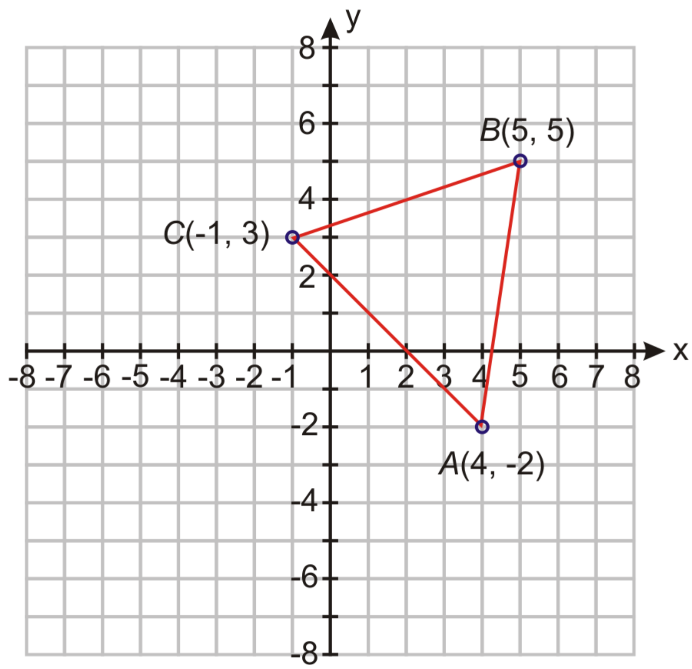
The distance and midpoint formula are applicable in geometry situations where we desire to find the distance between two points of the point halfway between two points.

Example 9

Plot the points $A = (4, -2), B = (5, 5)$, *and* $C = (-1, 3)$ *and connect them to make a triangle. Show* $\triangle ABC$ *is isosceles.*

Solution

Let's start by plotting the three points on the coordinate plane and making a triangle.



We use the distance formula three times to find the lengths of the three sides of the triangle.

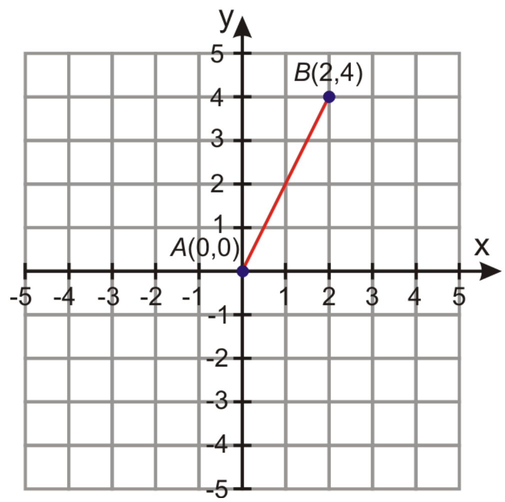
$$AB = \sqrt{(4-5)^2+(-2-5)^2} = \sqrt{(-1)^2+(-7)^2} = \sqrt{1+49} = \sqrt{50} = 5\sqrt{2}$$
$$BC = \sqrt{(5+1)^2+(5-3)^2} = \sqrt{(6)^2+(2)^2} = \sqrt{36+4} = \sqrt{40} = 2\sqrt{10}$$
$$AC = \sqrt{(4+1)^2+(-2-3)^2} = \sqrt{(5)^2+(-5)^2} = \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}$$

Notice that $AB = AC$, therefore $\triangle ABC$ is isosceles.

Example 10

At 8 AM one day, Amir decides to walk in a straight line on the beach. After two hours of making no turns and traveling at a steady rate, Amir was two mile east and four miles north of his starting point. How far did Amir walk and what was his walking speed?

Solution



Let's start by plotting Amir's route on a coordinate graph. We can place his starting point at the origin $A = (0,0)$. Then, his ending point will be at point $B = (2,4)$. The distance can be found with the distance formula.

$$d = \sqrt{(2-0)^2+(4-0)^2} = \sqrt{(2)^2+(4)^2} = \sqrt{4+16} = \sqrt{20}$$
$$d = 4.47 \ miles.$$

Since Amir walked 4.47 miles in 2 hours, his speed is

$$\text{Speed} = \frac{4.47 \ miles}{2 \ hours} = 2.24 \ mi/h$$

Review Questions

Find the distance between the two points.

1. (3, -4) and (6, 0)
2. (-1, 0) and (4, 2)
3. (-3, 2) and (6, 2)
4. (0.5, -2.5) and (4, -4)
5. (12, -10) and (0, -6)
6. (2.3, 4.5) and (-3.4, -5.2)
7. Find all points having an x coordinate of -4 and whose distance from point (4, 2) is 10.
8. Find all points having a y coordinate of 3 and whose distance from point (-2, 5) is 8.

Find the midpoint of the line segment joining the two points.

9. (3, -4) and (6, 1)
10. (2, -3) and (2, 4)
11. (4, -5) and (8, 2)
12. (1.8, -3.4) and (-0.4, 1.4)
13. (5, -1) and (-4, 0)

14. (10, 2) and (2, -4)
15. An endpoint of a line segment is (4, 5) and the midpoint of the line segment is (3, -2). Find the other endpoint.
16. An endpoint of a line segment is (-10, -2) and the midpoint of the line segment is (0, 4). Find the other endpoint.
17. Plot the points $A = (1,0), B = (6,4), C = (9,-2)$ and $D = (-6,-4), E = (-1,0), F = (2,-6)$. Prove that triangles ABC and DEF are congruent.
18. Plot the points $A = (4,-3), B = (3,4), C = (-2,-1), D = (-1,-8)$. Show that $ABCD$ is a rhombus (all sides are equal).
19. Plot points $A = (-5,3), B = (6,0), C = (5,5)$. Find the length of each side. Show that it is a right triangle. Find the area.
20. Find the area of the circle with center (-5, 4) and the point on the circle (3, 2).
21. Michelle decides to ride her bike one day. First she rides her bike due south for 12 miles, then the direction of the bike trail changes and she rides in the new direction for a while longer. When she stops, Michelle is 2 miles south and 10 miles west from her starting point. Find the total distance that Michelle covered from her starting point.

Review Answers

1. 5
2. $\sqrt{29}$
3. 9
4. 3.81
5. $4\sqrt{10}$
6. 11.25
7. (-4, -4) and (-4, 8)
8. (-9.75, 3) and (5.75, 3)
9. (4.5, -1.5)
10. (2, 0.5)
11. (6, -1.5)
12. (.7, -1)
13. (0.5, -0.5)
14. (6, -1)
15. (2, -9)
16. (10, 10)
17. $AB = DE = 6.4, AC = DF = 8.25, BC = EF = 6.71$
18. $AB = BC = CD = DA = 7.07$
19. $AB = \sqrt{130}, AC = \sqrt{104}, BC = \sqrt{26}$ and $\left(\sqrt{26}\right)^2 + \left(\sqrt{104}\right)^2 = \left(\sqrt{130}\right)^2$. Right triangle.
20. Radius $= 2\sqrt{17}$, Area $= 68\pi$
21. 26.14 miles

2.7 Imaginary and Complex Numbers

Learning Objectives

- Write square roots with negative radicands in terms of i
- Recognize and write complex numbers in standard form
- Describe the relationship between the sets of integers, rational numbers, real numbers and complex numbers
- Plot $z = a + bi$ in the complex number plane

Introduction

While working with quadratic equations, you may have solved an equation such as:

$(x-1)^2+4=0.$

No matter which method of solving quadratics you used, the solutions to that equation are not real numbers, and you find that they are 1 + 2i and 1 - 2i. (Recall that $\sqrt{-1} = i$). These solutions combine **imaginary** and **real** numbers, and are called **complex** numbers.

The use of the word *imaginary* does not mean these numbers are useless. For a long period in the history of mathematics, it was thought that the square root of a negative number was in fact only within the mathematical imagination without real-world significance hence, imaginary. That has changed. Mathematicians now consider the imaginary number as another set of numbers that have real significance, but do not fit on what is called the number line—and engineers, scientists, and others solve real world problems using complex numbers!

Recognize

Recognize $i = \sqrt{-1}$, $\sqrt{-x} = i\sqrt{x}$

Where did complex numbers come from? If you solve the equation $x^2+1=0$, you get $x = \pm\sqrt{-1}$. But there is no real number that, when multiplied by itself, yields -1. To fix this problem, mathematicians defined the imaginary constant, i. by definition,

$i = \sqrt{-1}$

or squaring both sides,

$i^2 = -1$

Recall that you can simplify radicals by factoring out perfect squares in the radicand. For instance, $\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$. The same procedure works with i. If you have a negative number in the radicand, you can factor out the -1 and use the identity $i = \sqrt{-1}$ to simplify.

Example: Simplify $\sqrt{-5}$

Solution:

$\sqrt{-5} = \sqrt{(-1)\cdot(5)}$

$= \sqrt{-1}\sqrt{5}$

$= i\sqrt{5}$

This also works in combination with the other method of factoring out perfect squares. See the following example.

Example: Simplify $\sqrt{-72}$

Solution:

$\sqrt{-72} = \sqrt{(-1)\cdot(72)}$

$= \sqrt{-1}\sqrt{72}$

$= i\sqrt{72}$

But, we're not done yet. 72 = 36·2, so

$i\sqrt{72} = i\sqrt{36}\sqrt{2}$

$= i(6)\sqrt{2}$

$= 6i\sqrt{2}$

Standard Form of Complex Numbers (a bi)

Sometimes when you solve a quadratic equation, the solution has both a real part and an imaginary part. For example, if you want to solve

$(x-1)^2+4=0$

then

$(x-1)^2=-4$

$x-1=\pm\sqrt{-4}$

$x-1=\pm\sqrt{-1}\sqrt{4}$

$x-1=\pm 2i$

$x=1\pm 2i$

$x=1\pm 2i \; or \; 1-2i$

Notice that these two solutions involve a real part, 1, and an imaginary part, $\pm 2i$

$z = a + bi$ is the **standard** or **rectangular** form of a complex number.

A complex number is a number that has a real part (in this case a) and an imaginary part, that is, the imaginary number i with a coefficient b.

Set of Complex Numbers (complex, real, irrational, rational, etc.)

The complex numbers are a superset of the real numbers. Given $z = a + bi$, if $b = 0$ then z is a real number. Every real number can be written as a complex number (just let it equal a), but there are many more complex numbers than real numbers. Hence the complex numbers are a superset of the real numbers.

When you were first introduced to mathematics, you probably used **positive whole numbers**, that is 1, 2, 3, 4,. . .

Later, negative whole numbers are investigated. The set of all whole numbers, both positive and negative, including the number zero, is known as **integers**: . . . - 2, - 1, 0, 1, 2,. . .

Later, students are introduced to fractions. The set of all numbers that CAN be expressed as a quotient of two integers (where the denominator is not zero) is called the set of **Rational Numbers**. Rational Numbers can also be expressed as a terminating or repeating decimal. Some rational numbers are $-1, \frac{3}{5}, -\frac{7}{3}, 1,000,002, 0$. Of course there are an infinite number of rational numbers between any two whole numbers, so listing all rational numbers neatly is difficult (but it is possible–can you think of a way to do it?).

Notice, that all integers are in the set of rational numbers (for example, 5 CAN be written as the quotient of 10 and

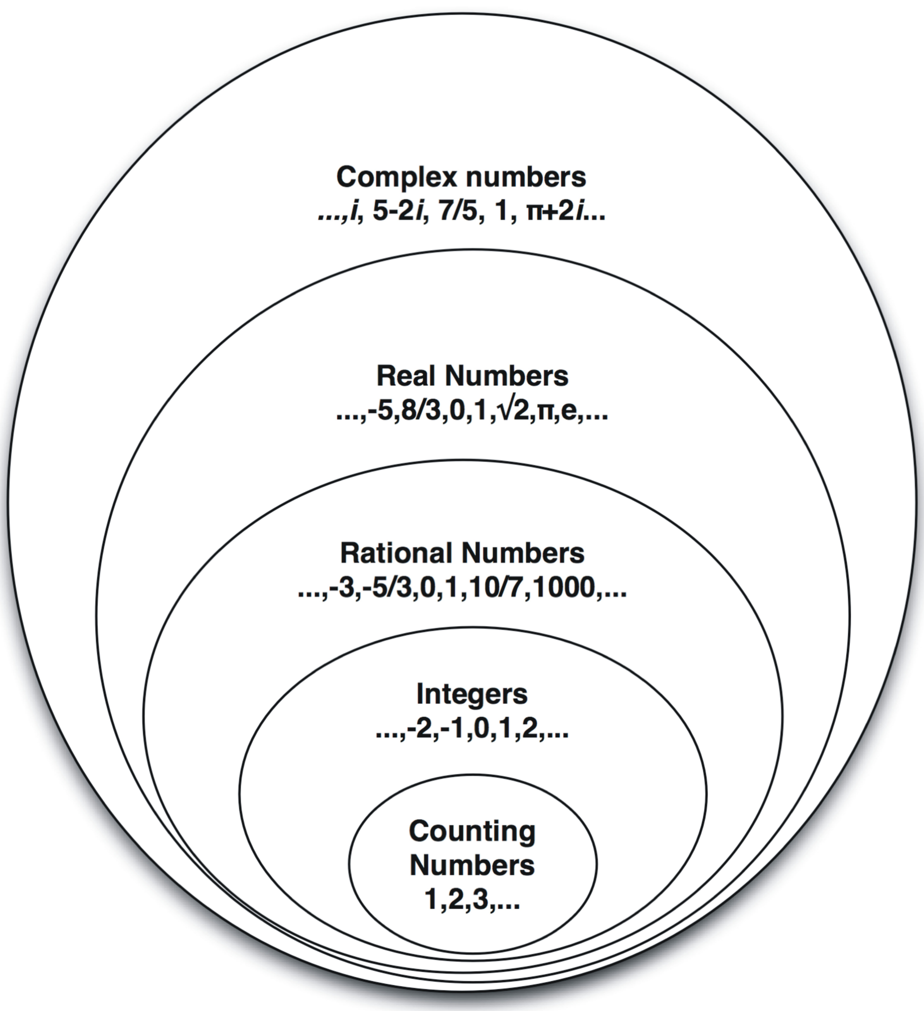
2 since $5 = \frac{10}{2}$), so the integers are a subset of the rational numbers. Finally, when working with circles students encounter a number that can be approximated as a quotient of two integers but cannot be expressed EXACTLY as that quotient, that is the number π. Recall that π was often expressed as APPROXIMATELY $\frac{22}{7}$ or 3.14, BUT NOT EXACTLY THOSE VALUES.

When first exploring using the Pythagorean Theorem to find the length of a diagonal of a square whose side is 1, the number $\sqrt{2}$ was introduced. $\sqrt{2}$ often was approximated as 1.4 or 1.414, but again you can't possibly write out all of the decimals in $\sqrt{2}$. These two numbers are examples of IRRATIONAL numbers, that is numbers that cannot be expressed as a quotient of two integers, and therefore CANNOT be expressed as a terminating or repeating decimal. The set of all rational and irrational numbers together is called REAL numbers.

Finally, when working again with the Pythagorean Theorem in the coordinate plane (where "negative distances" are possible), negative values appeared within the square root! But what number times itself can result in a negative number?

Historically, when this occurred, mathematicians thought that this was only an oddity of the theorem and not something that can actually exist. They therefore called such numbers imaginary. But, some real-world problems can be solved with imaginary numbers.

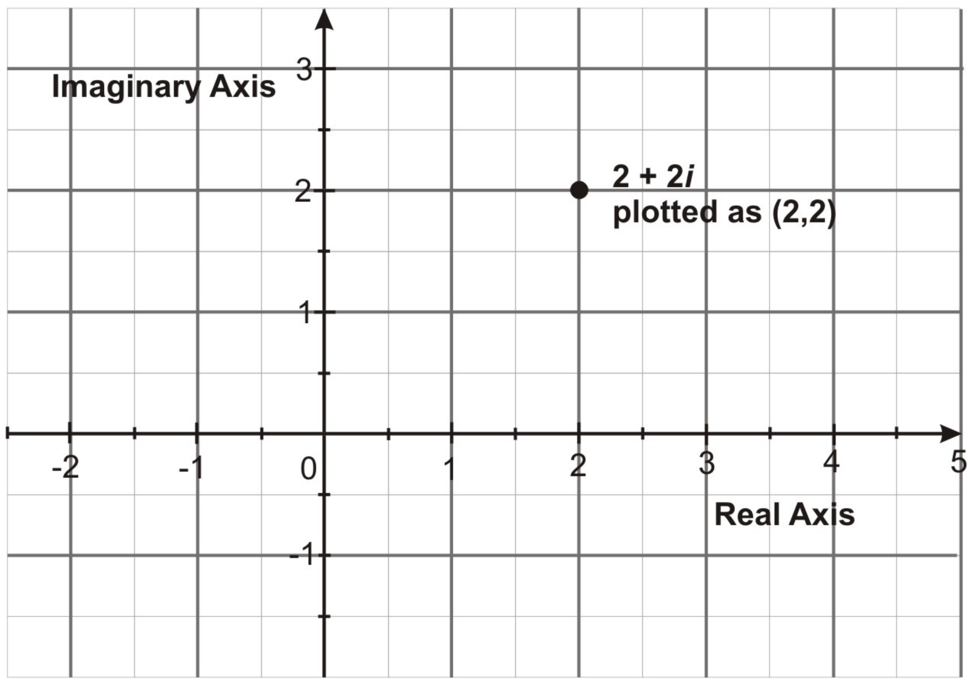
The set of all real numbers AND imaginary numbers is called the set of Complex numbers.



Complex Number Plane

In standard form $z = a + bi$, a complex number can be graphed using rectangular coordinates (a, b). a represents the x - coordinate, while b represents the y - coordinate. Alternatively, the x - coordinate represents "real number" values, while the$y$ - coordinate represents the "imaginary" values.

For example, given the complex number in standard form: $z = 2 + 2i$, you can graph this number in the coordinate plane To graph this point, the coordinate (2, 2) is graphed as shown below:



Lesson Summary

When graphing a complex number using rectangular coordinates, the x-axis plots the real number, while the y-axis plots the coefficient of the imaginary number.

Points to Consider

Given a point in a rectangular coordinate system that represents a complex number, multiply that complex number by i and graph this new complex number. If the points that represent the original complex number and the new complex number have a line drawn from the origin to each point- note the angle between the two lines. Multiply the second complex number by i and plot this third point. Draw a line from the origin to this point. Note the angle between the second line and the third line. What appears to happen when a complex number is multiplied by i?

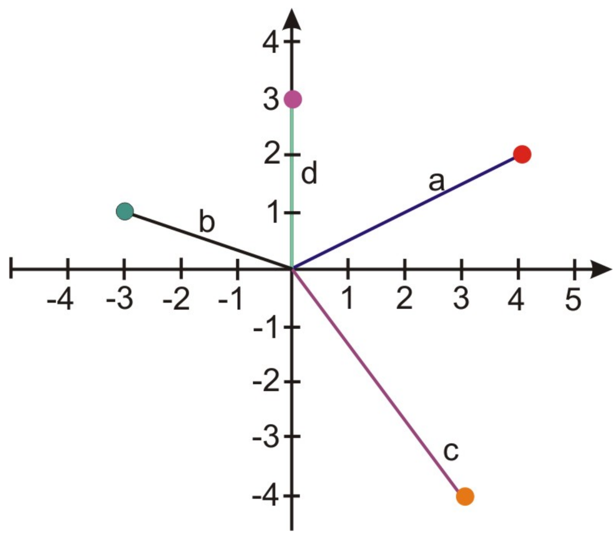
Review Questions

Using rectangular coordinate system, graph

1. Simplify the following radicals
   a. $\sqrt{-9}$
   b. $\sqrt{-12}$
   c. $\sqrt{-17}$
   d. $\sqrt{140-108}$

2. Solve each equation and express it as a complex number. (Note: If the imaginary part is 0, you can still express the solution as $a + bi$, but you will have $b = 0$
   a. $x^2 + 24 = 0$
   b. $2x^2 - 4x + 7 = 0$
3. Plot each of the following complex numbers:
   a. $(4 + 2i)$
   b. $(-3 + i)$
   c. $(3 - 4i)$
   d. $3i$

Review Answers

1. a. $3i$
   b. $2i\sqrt{3}$
   c. $i\sqrt{17}$
   d. $\sqrt{32} = 6\sqrt{2}$
2. a. $x = \pm 2\sqrt{6}i$
   b. $x = 1 \pm 2\sqrt{3}i$
3. 

2.8 Operations on Complex Numbers

Learning Objectives

- Add and subtract complex numbers
- Find the complex conjugate of a complex number
- Multiply and divide complex numbers
- Use your calculator to add, subtract, multiply, and divide complex numbers

Sums and Differences of Complex Numbers

Recall that a complex number is given in the form $a + bi$ where a and b are real numbers and i is the imaginary constant, $i^2 = -1$.

When adding (or subtracting) two or more complex numbers the fastest method will be to add (or subtract) the real components to obtain the sum of the real numbers, and then separately add (or subtract) the imaginary coefficients to obtain the sum of the imaginary numbers or:

$(a + bi) + (c + di) = [a + c] + [b + d]i$

Example: Perform the Indicated Operation:

$(6 + 3i) + (-5 + 2i) = (6 + -5) + (3 + 2)\ i = 1 + 5i$

$(3 - 2i) - (2 - 4i) = (3 - 2) + (-2 - (-4))\ i = 1 + 2i$

$(6) + (4 - 3i) = (6 + 4) + (0 + (-3))\ i = 10 - 3i$

Products and Quotients of Complex Numbers (conjugates)

Multiplying Complex Numbers:

When multiplying complex numbers in rectangular form, recall the method for multiplying two binomials (sometimes called FOIL): $(m + n)(x + y) = mx + my + nx + ny$. We use the same procedure for multiplying complex numbers:

$(a + bi) + (c + di) = ac + adi + bci + bdi^2$

But, unlike the algebraic expression, the above expression contains the number, i.

Recall that $i^2 = -1$, so $bdi^2 = bd(-1) = -bd$ and

$adi + bci$ can be combined and then factored as $(ad + bc)i$. Thus we have the general result,

$(a + bi) + (c + di) = (ac - bd) + (ad + bd)i$

Example: Multiply: $(6 + 3i)(2 - 3i)$

$(6+3i)(2-3i) = 12-18i+6i-9i^2$

$= 21-12i$

(Combining like terms $-9i^2$ reduces to -9(-1) or 9)

Example: Multiply: $(5 - 7i)(5 + 7i)$

$(5-7i)(5+7i) = 25+35i-35i-49i^2$

or

$= 25+0-49(-1)$

$= 74$

Note: The product of the complex number (5 - 7i) and the complex number (5 + 7i) is a real number. When a complex number is multiplied by another complex number to produce a real number, the two complex numbers are called **complex conjugates.**

Example: Multiply $i(6 - 2i)$

$6i-2i^2 = 6i-2(-1)$

$= 6i+2$

But, the real part of a complex number is generally written first, so we can write this as

$= 2+6i$

Dividing Complex Numbers

To divide two complex numbers is similar to dividing two irrational numbers. Recall that in that problem, the procedure was to find the irrational **conjugate** of the denominator and then multiply both the numerator and the denominator by this conjugate

Divide: $\frac{3}{1+\sqrt{2}}$

First find the irrational conjugate of the denominator: $1-\sqrt{2}$, then multiply both the numerator and the denominator by this value:

$\frac{3}{1+\sqrt{2}} \times \frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{3-3\sqrt{2}}{1-2}$

this reduces to

$= \frac{3-3\sqrt{2}}{-1}$

or

$= -3+3\sqrt{2}$

In this case since you are interested in eliminating the complex numbers from the denominator, find the **complex conjugate** of the denominator and multiply BOTH the numerator AND the denominator by it.

You find the complex conjugate in the same way you found the conjugate of irrational numbers, change the sign of the imaginary part. For instance, the complex conjugate of 4 + 3i is 4 – 3i

A complex number multiplied by its complex conjugate will yield a real number. By recalling $(a + b)(a - b) = a^2 - b^2$ the complex conjugate can be found:

> The conjugate of 4 + 3i is found by retaining the real part (4) and reversing only the sign of the imaginary part (that is, 3i becomes -3i) $(4 + 3i)(4 - 3i) = 16 - 12i + 12i - 9i^2$ Notice that $-12i$ and $12i$ cancel. Also recall that $i^2 = -1$ 16 + 9 = 25 (4 + 3i)(4 – 3i) = 25

The product of this complex number and its conjugate is 25.

When multiplying complex numbers sometimes intuition about the nature of the product can mislead.

For example in (a + b)(a + b), where all of the terms are real numbers, no terms of each of the four products will cancel. Some of the terms may be combined. However in (1 + i)(1 - i), where some terms are real numbers and some terms are imaginary numbers, this is no longer true. Two of these terms cancel: the first product yields 1 while the last product yields i^2 or -1, and those terms cancel!

Example: Find the quotient: $\frac{6-3i}{4+3i}$

First, observe that the complex conjugate of the denominator is 4 – 3i

Multiply both the numerator and the denominator by 4 – 3i:

$\frac{6-3i}{4+3i} \times \frac{4-3i}{4-3i} = \frac{24-18i+12i+9i^2}{16-12i+12i-9i^2}$

$= \frac{15-6i}{25}$

$\frac{6-3i}{4+3i} = \frac{15-6i}{25}$

This number can also be written as $\frac{15}{25} - \frac{6i}{25}$ or 0.6 – 0.24 *i*

Applications, Technological Tools

A graphing calculator can perform operations with complex numbers. Press mode. Scroll down until: real *a* + *bi re*^ θ*i* is seen, then select *a* + *bi*. Press Quit. Now the calculator is able to perform operations with complex numbers in *a* + *bi* form.

When the calculator is in complex number mode, be sure to use parenthesis to group the parts of the complex numbers. For example, enter 1 + 3*i* as (1 + 3*i*). Try doing some of the calculations from this section on your calculator to verify that complex mode works.

Lesson Summary

When adding and subtracting complex numbers add/subtract the real numbers and then add/subtract the imaginary numbers.

When multiplying complex numbers use the FOIL method of multiplication. Be sure to substitute i^2 = -1 when appropriate.

When dividing complex numbers, write the problem as a fraction and then multiply both numerator and denominator by the conjugate of the denominator. If this process is done successfully, there will be only a real number in the denominator.

Points to Consider

Complex numbers- that is numbers that have a real and an imaginary part, also known as components, have characteristics similar to working with two unlike terms. However, they also have major differences such as not having as a final answer to a question, a denominator with an imaginary component or being sure to reduce i^2 to -1.

Review Questions

Perform the indicated operations:

1. a. (-8 - 2*i*) + (5 - 2*i*)
   b. (6 + *i*) - (5 - 2*i*)
   c. (-3*i*) - (2 - 3*i*)
   d. (3 + 2*i*)(5 - *i*)
   e. (0 + 4*i*)(3 - 1)
   f. $(5 - 3i)^2$
   g. $\frac{6-i}{4+i}$

h. (5 - 2i) ÷ (-2 + 3i)

Review Answers

1. a. (3 - 4i)
   b. (1 + 3i)
   c. (-2 + 0i)
   d. (17 + 7i)
   e. (4 + 12i)
   f. (16 - 30i)
   g. $\frac{23-10i}{17}$
   h. $\frac{-16-11i}{13}$

CHAPTER 3 Quadratic Equations and Quadratic Functions

Chapter Outline

3.1 Graphs of Quadratic Functions

Learning Objectives

- Graph quadratic functions.
- Compare graphs of quadratic functions.
- Graph quadratic functions in intercept form.
- Analyze graphs of real-world quadratic functions.

Introduction

The graphs of quadratic functions are curved lines called **parabolas**. You don't have to look hard to find parabolic shapes around you. Here are a few examples.

- The path that a ball or a rocket takes through the air.
- Water flowing out of a drinking fountain.
- The shape of a satellite dish.
- The shape of the mirror in car headlights or a flashlight.

Graph Quadratic Functions

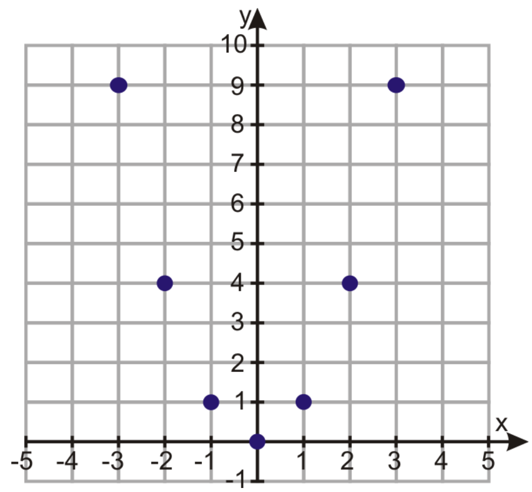
Let's see what a parabola looks like by graphing the simplest quadratic function, $y = x^2$.

We will graph this function by making a table of values. Since the graph will be curved we need to make sure that we pick enough points to get an accurate graph.

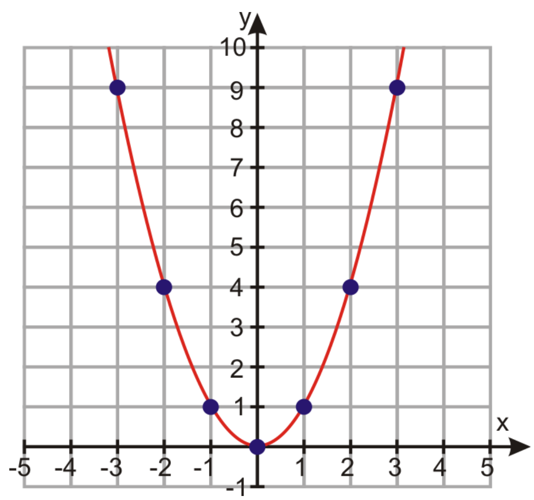
TABLE 3.1:

x	$y = x^2$
-3	$(-3)^2 = 9$
-2	$(-2)^2 = 4$
-1	$(-1)^2 = 1$
0	$(0)^2 = 0$
1	$(1)^2 = 1$
2	$(2)^2 = 4$
3	$(3)^2 = 9$

We plot these points on a coordinate graph.



To draw the parabola, draw a smooth curve through all the points. (Do not connect the points with straight lines).



Let's graph a few more examples.

Example 1

Graph the following parabolas.

a) $y = 2x^2 + 4x + 1$

b) $y = -x^2$

c) $y = x^2 - 2x + 3$

Solution

a) $y = 2x^2 + 4x + 1$

Make a table of values.

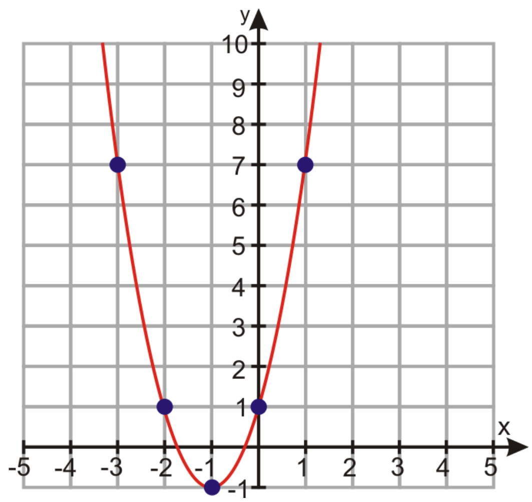
TABLE 3.2:

x	$y = 2x^2 + 4x + 1$
-3	$2(-3)^2 + 4(-3) + 1 = 7$
-2	$2(-2)^2 + 4(-2) + 1 = 1$
-1	$2(-1)^2 + 4(-1) + 1 = -1$
0	$2(0)^2 + 4(0) + 1 = 1$

TABLE 3.2: (continued)

x	$y = 2x^2 + 4x + 1$
1	$2(1)^2 + 4(1) + 1 = 7$
2	$2(2)^2 + 4(2) + 1 = 17$
3	$2(3)^2 + 4(3) + 1 = 31$

Notice that the last two points have large $y-$values. We will not graph them since that will make our $y-$scale too big. Now plot the remaining points and join them with a smooth curve.



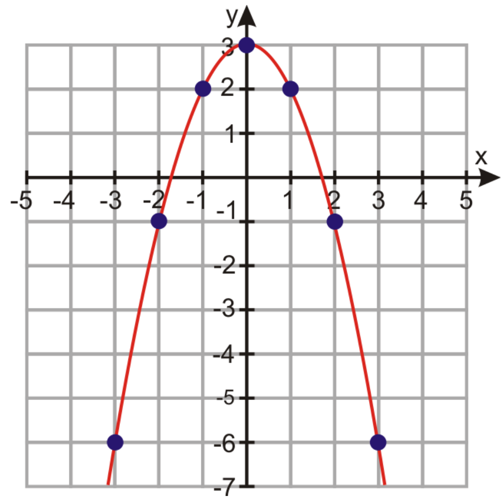
b) $y = -x^2 + 3$

Make a table of values.

TABLE 3.3:

x	$y = -x^2 + 3$
-3	$-(-3)^2 + 3 = -6$
-2	$-(-2)^2 + 3 = -1$
-1	$-(-1)^2 + 3 = 2$
0	$-(0)^2 + 3 = 3$
1	$-(1)^2 + 3 = 2$
2	$-(2)^2 + 3 = -1$
3	$-(3)^2 + 3 = -6$

Plot the points and join them with a smooth curve.



Notice that it makes an "upside down" parabola. Our equation has a negative sign in front of the x^2 term. The sign of the coefficient of the x^2 term determines whether the parabola turns up or down.

If the coefficient of x^2; is positive, then the parabola turns up.

If the coefficient of x^2; is negative, then the parabola turns down.

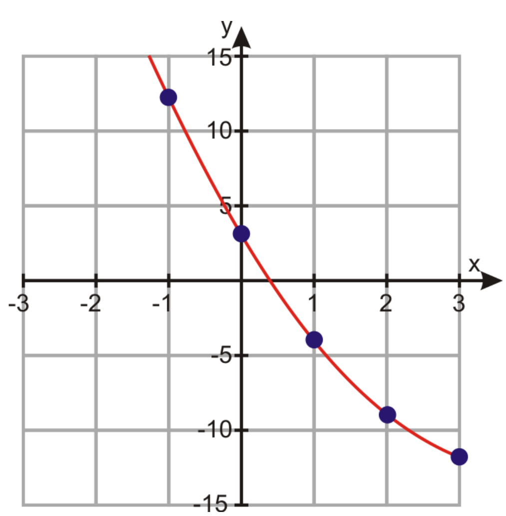
c) $y = x^2 - 8x + 3$

Make a table of values.

TABLE 3.4:

x	$y = x^2 - 8x + 3$
-3	$(-3)^2 - 8(-3) + 3 = 36$
-2	$(-2)^2 - 8(-2) + 3 = 23$
-1	$(-1)^2 - 8(-1) + 3 = 12$
0	$(0)^2 - 8(0) + 3 = 3$
1	$(1)^2 - 8(1) + 3 = -4$
2	$(2)^2 - 8(2) + 3 = -9$
3	$(3)^2 - 8(3) + 3 = -12$

Let's not graph the first two points in the table since the values are very big. Plot the points and join them with a smooth curve.

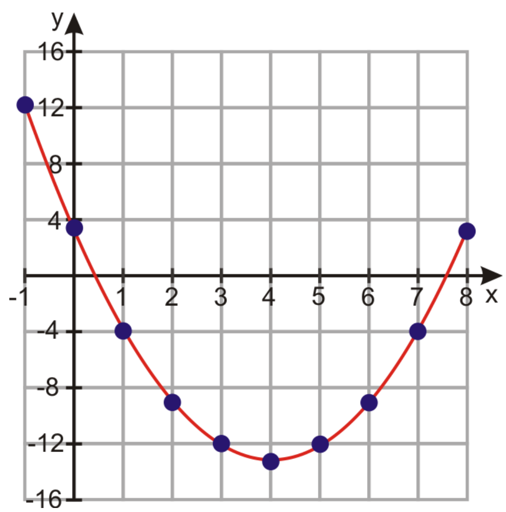


This does not look like the graph of a parabola. What is happening here? If it is not clear what the graph looks like choose more points to graph until you can see a familiar curve. For negative values of x it looks like the values of y are getting bigger and bigger. Let's pick more positive values of x beyond $x = 3$.

TABLE 3.5:

x	$y = x^2 - 8x + 3$
-1	$(-1)^2 - 8(-1) + 3 = 12$
0	$(0)^2 - 8(0) + 3 = 3$
1	$(1)^2 - 8(1) + 3 = -4$
0	$(0)^2 - 8(0) + 3 = 3$
1	$(1)^2 - 8(1) + 3 = -4$
2	$(2)^2 - 8(2) + 3 = -9$
3	$(3)^2 - 8(3) + 3 = -12$
4	$(4)^2 - 8(4) + 3 = -13$
5	$(5)^2 - 8(5) + 3 = -12$
6	$(6)^2 - 8(6) + 3 = -9$
7	$(7)^2 - 8(7) + 3 = -4$
8	$(8)^2 - 8(8) + 3 = 3$

Plot the points again and join them with a smooth curve.



We now see the familiar parabolic shape. Graphing by making a table of values can be very tedious, especially in problems like this example. In the next few sections, we will learn some techniques that will simplify this process greatly, but first we need to learn more about the properties of parabolas.

Compare Graphs of Quadratic Functions

The **general form** (or **standard form**) of a quadratic function is:

$$y = ax^2 + bx + c$$

Here a, b and c are the **coefficients.** Remember a coefficient is just a number (i.e. a constant term) that goes before a variable or it can be alone. You should know that if you have a quadratic function, its graph is always a parabola.

While the graph of a quadratic is always the same basic shape, we have different situations where the graph could be upside down. It could be shifted to different locations or it could be "fatter" or "skinnier". These situations are determined by the values of the coefficients. Let's see how changing the coefficient changes the orientation, location or shape of the parabola.

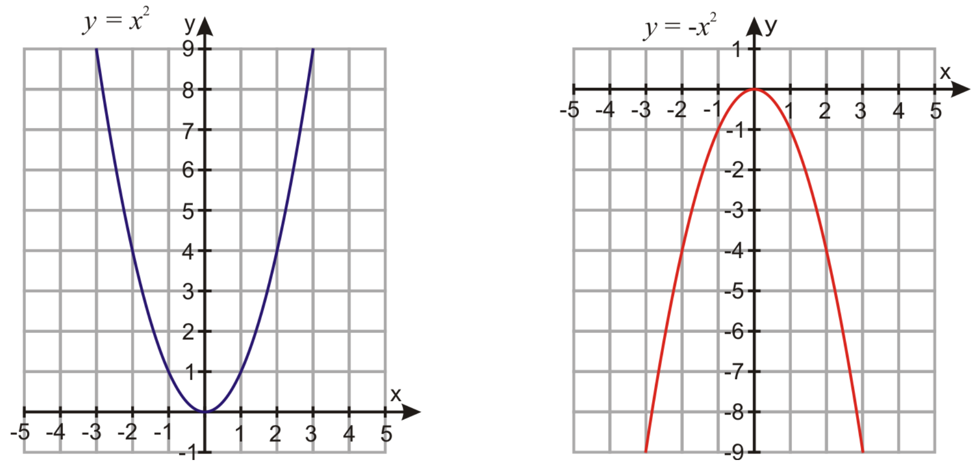
Orientation

Does the parabola open up or down?

The answer to that question is pretty simple:

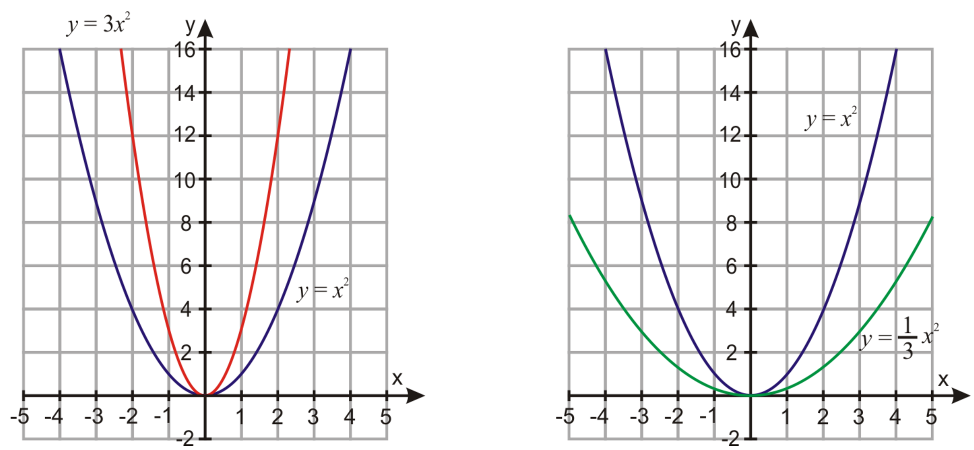
- If a is positive, the parabola opens up.
- If a is negative, the parabola opens down.

The following plot shows the graphs of $y = x^2$ and $y = -x^2$. You see that the parabola has the same shape in both graphs, but the graph of $y = x^2$ is right-side-up and the graph of $y = -x^2$ is upside-down.



Dilation

Changing the value of the coefficient a makes the graph "fatter" or "skinnier". Let's look at how graphs compare for different positive values of a. The plot on the left shows the graphs of $y = -x^2$ and $y = 3x^2$. The plot on the right shows the graphs of $y = -x^2$ and $y = \left(\frac{1}{3}\right)x^2$.



Notice that the larger the value of a is, the skinnier the graph is. For example, in the first plot, the graph of $y = 3x^2$ is skinnier than the graph of $y = x^2$. Also, the smaller a is (i.e. the closer to 0), the fatter the graph is. For example, in the second plot, the graph of $y = \left(\frac{1}{3}\right)x^2$ is fatter than the graph of $y = x^2$. This might seem counter-intuitive, but if you think about it, it should make sense. Let's look at a table of values of these graphs and see if we can explain why this happens.

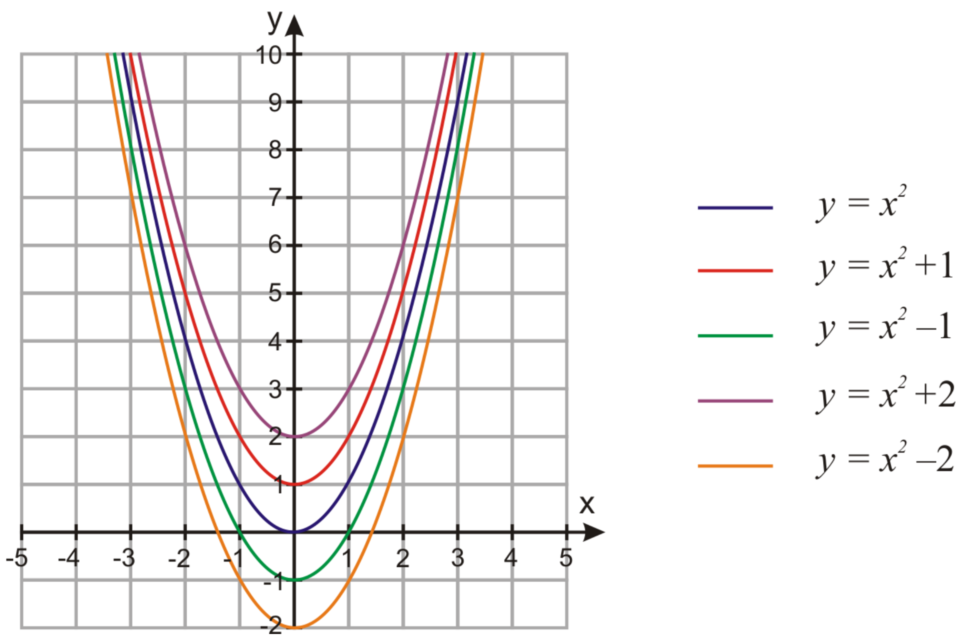
TABLE 3.6:

x	$y = x^2$	$y = 3x^2$	$y = \frac{1}{3}x^2$
-3	$(-3)^2 = 9$	$3(-3)^2 = 27$	$(-3)^{\frac{2}{3}} = 3$
-2	$(-2)^2 = 4$	$3(-2)^2 = 12$	$(-2)^{\frac{2}{3}} = \frac{4}{3}$
-1	$(-1)^2 = 1$	$3(-1)^2 = 3$	$(-1)^{\frac{2}{3}} = \frac{1}{3}$
0	$(0)^2 = 0$	$3(0)^2 = 0$	$(0)^{\frac{2}{3}} = 0$
1	$(1)^2 = 1$	$3(1)^2 = 3$	$(1)^{\frac{2}{3}} = \frac{1}{3}$
2	$(2)^2 = 4$	$3(2)^2 = 12$	$(2)^{\frac{2}{3}} = \frac{4}{3}$
3	$(3)^2 = 9$	$3(3)^2 = 27$	$(3)^{\frac{2}{3}} = 3$

From the table, you can see that the values of $y = 3x^2$ are bigger than the values of $y = x^2$. This is because each value of y gets multiplied by 3. As a result, the parabola will be skinnier because it grows three times faster than $y = x^2$. On the other hand, you can see that the values of $y = \left(\frac{1}{3}\right)x^2$ are smaller than the values of $y = x^2$. This is because each value of y gets divided by 3. As a result, the parabola will be fatter because it grows at one third the rate of $y = x^2$.

Vertical Shift

Changing the value of the coefficient c (called the constant term) has the effect of moving the parabola up and down. The following plot shows the graphs of $y = x^2, y = x^2 + 1, y = x^2 - 1, y = x^2 + 2, y = x^2 - 2$.



We see that if c is positive, the graph moves up by c units. If c is negative, the graph moves down by c units. In one of the later sections we will also talk about **horizontal shift (i.e. moving to the right or to the left).** Before we can do that we need to learn how to rewrite the quadratic equations in different forms.

Graph Quadratic Functions in Intercept Form

As you saw, in order to get a good graph of a parabola, we sometimes need to pick a lot of points in our table of values. Now, we will talk about different properties of a parabola that will make the graphing process less tedious. Let's look at the graph of $y = x^2 - 6x + 8$.

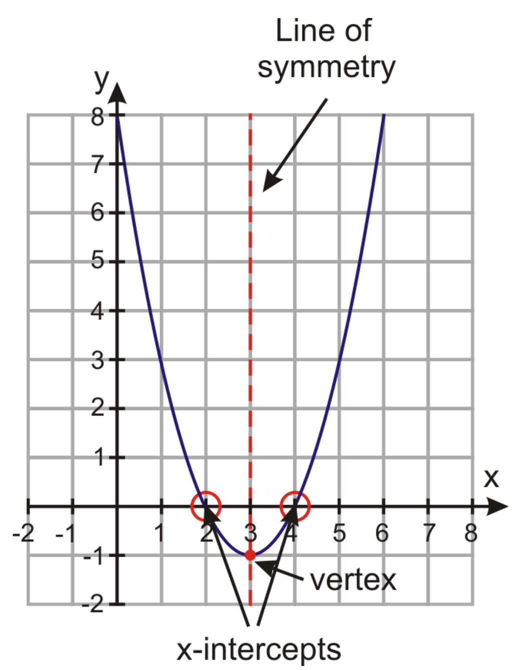
There are several things that we notice.

1. The parabola crosses the x axis at two points: $x = 2$ and $x = 4$.

- These points are called the $x-$**intercepts** of the parabola.

2. The lowest point of the parabola occurs at point (3, -1).

- This point is called the **vertex** of the parabola.
- The vertex is the lowest point in a parabola that turns upward, and it is the highest point in a parabola that turns downward.
- The vertex is exactly halfway between the two $x-$intercepts. This will always be the case and you can find the vertex using that rule.



3. A parabola is **symmetric.** If you draw a vertical line through the vertex, you can see that the two halves of the parabola are mirror images of each other. The vertical line is called the **line of symmetry.**

We said that the general form of a quadratic function is $y = ax^2 + bx + c$. If we can factor the quadratic expression, we can rewrite the function in **intercept form**

$$y = a(x - m)(x - n)$$

This form is very useful because it makes it easy for us to find the $x-$intercepts and the vertex of the parabola. The $x-$intercepts are the values of x where the graph crosses the $x-$axis. In other words, they are the values of x when $y = 0$. To find the $x-$intercepts from the quadratic function, we set $y = 0$ and solve.

$$0 = a(x-m)(x-n)$$

Since the equation is already factored, we use the zero-product property to set each factor equal to zero and solve the individual linear equations.

$$x-m=0 \qquad \text{or} \qquad x-n=0$$
$$x=m \qquad\qquad x=n$$

So the $x-$intercepts are at points $(m,0)$ and $(n,0)$.

Once we find the $x-$intercepts, it is simple to find the vertex. The $x-$coordinate of the vertex is halfway between the two x intercepts, so we can find it by taking the average of the two values $\frac{(m+n)}{2}$.

The $y-$value can be found by substituting the value of x back into the equation of the function.

Let's do some examples that find the $x-$intercepts and the vertex:

Example 2

Find the $x-$ intercepts and the vertex of the following quadratic function.

(a) $y = x^2 - 8x + 15$

(b) $y = 3x^2 + 6x - 24$

Solution

a) $y = x^2 - 8x + 15$

Write the quadratic function in intercept form by factoring the right hand side of the equation.

Remember, to factor the trinomial we need two numbers whose product is 15 and whose sum is -8. These numbers are -5 and -3.

The function in intercept form is $y = (x-5)(x-3)$

We find the $x-$intercepts by setting $y = 0$.

We have

$$0 = (x-5)(x-3)$$
$$x-5=0 \qquad \text{or} \qquad x-3=0$$
$$x=5 \qquad\qquad x=3$$

The $x-$intercepts are (5, 0) and (3, 0).

The vertex is halfway between the two $x-$intercepts. We find the x value by taking the average of the two $x-$intercepts, $x = \frac{(5+3)}{2} = 4$.

We find the y value by substituting the x value we just found back into the original equation.

$$y = x^2 - 8x + 15 \Rightarrow y = (4)^2 - 8(4) + 15 = 16 - 32 + 15 = -1$$

The vertex is (4, -1).

b) $y = 3x^2 + 6x - 24$

Rewrite the function in intercept form.

Factor the common term of 3 first $y = 3(x^2 + 2x - 8)$.

Then factor completely $y = 3(x+4)(x-2)$

Set $y = 0$ and solve: $0 = 3(x+4)(x-2)$.

$$x + 4 = 0 \Rightarrow x = -4$$

or

$$x - 2 = 0 \Rightarrow x = 2$$

The $x-$**intercepts are**: (-4, 0) **and** (2, 0)

For the vertex,

$x = \frac{-4+2}{2} = -1$ and $y = 3(-1)^2 + 6(-1) - 24 = 3 - 6 - 24 = -27$

The vertex is: (-1, -27).

When graphing, it is very useful to know the vertex and $x-$intercepts. Knowing the vertex, tells us where the middle of the parabola is. When making a table of values we pick the vertex as a point in the table. Then we choose a few smaller and larger values of x. In this way, we get an accurate graph of the quadratic function without having to have too many points in our table.

Example 3

Find the x−intercepts and vertex. Use these points to create a table of values and graph each function.

a) $y = x^2 - 4$

b) $y = -x^2 + 14x - 48$

Solution

a) $y = x^2 - 4$

Let's find the $x-$intercepts and the vertex.

Factor the right-hand-side of the function to put the equation in intercept form.

$$y = (x-2)(x+2)$$

Set $y = 0$ and solve.

$$0 = (x-2)(x+2)$$

$$x - 2 = 0 \quad \text{or} \quad x + 2 = 0$$

$$x = 2 \qquad\qquad x = -2$$

$x-$intercepts are: (2, 0) and (-2, 0)

Find the vertex.

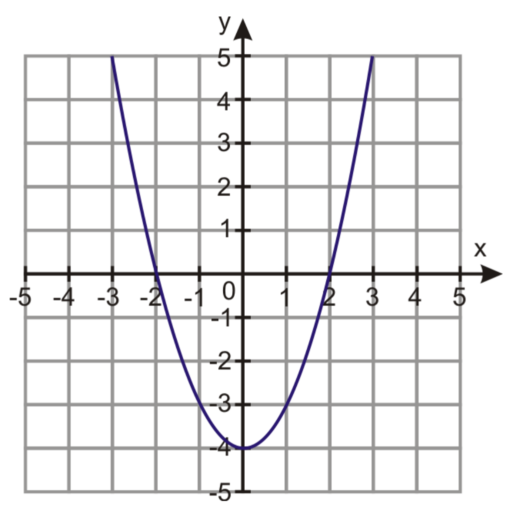
$$x = \frac{2-2}{2} = 0 \qquad y = (0)^2 - 4 = -4$$

The vertex is (0, -4)

Make a table of values using the vertex as the middle point. Pick a few values of x smaller and larger than $x = 0$. Include the $x-$intercepts in the table.

TABLE 3.7:

	x	$y = x^2 - 4$
	-3	$y = (-3)^2 - 4 = 5$
$x-$intercept	-2	$y = (-2)^2 - 4 = 0$
	-1	$y = (-1)^2 - 4 = -3$
vertex	0	$y = (0)^2 - 4 = -4$
	1	$y = (1)^2 - 4 = -3$
$x-$intercept	2	$y = (2)^2 - 4 = 0$
	3	$y = (3)^2 - 4 = 5$



b) $y = -x^2 + 14x - 48$

Let's find the $x-$intercepts and the vertex.

Factor the right hand side of the function to put the equation in intercept form.

$$y = -(x^2 - 14x + 48) = -(x-6)(x-8)$$

Set $y = 0$ and solve.

Set $y = 0$ and solve.

$$ath = 0 = -(x-6)(x-8)$$

$$x-6=0 \quad \text{or} \quad x-8=0$$

$$x=6 \qquad\qquad x=8$$

The $x-$intercepts are: (6, 0) and (8, 0)

Find the vertex

$$ath = x = \frac{6+8}{2} = 7 \qquad y = (7)^2 + 14(7) - 48 = 1$$

The vertex is (7, 1).

Make a table of values using the vertex as the middle point. Pick a few values of x smaller and larger than $x = 1$. Include the $x-$intercepts in the table. Then graph the parabola.

TABLE 3.8:

	x	$y = -x^2 + 14x - 48$
	4	$y = -(4)^2 + 14(4) - 48 = -8$
	5	$y = -(5)^2 + 14(5) - 48 = -3$
$x-$ intercept	6	$y = -(6)^2 + 14(6) - 48 = 0$
$x-$ vertex	7	$y = -(7)^2 + 14(7) - 48 = 1$
$x-$ intercept	8	$y = -(8)^2 + 14(8) - 48 = 0$
	9	$y = -(9)^2 + 14(9) - 48 = -3$
	10	$y = -(10)^2 + 14(10) - 48 = -8$

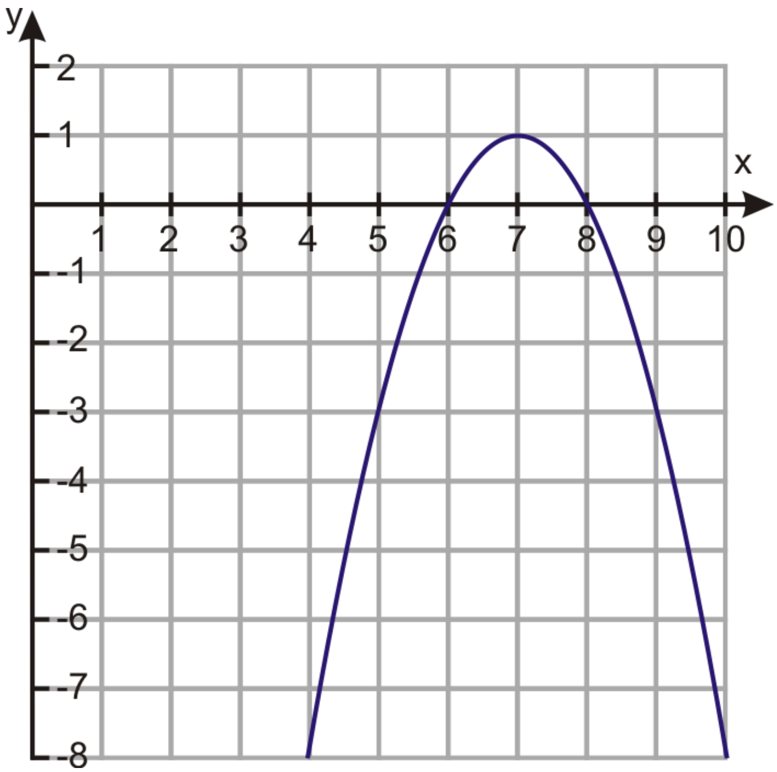


FIGURE 3.1

Analyze Graphs of Real-World Quadratic Functions.

As we mentioned at the beginning of this section, parabolic curves are common in real-world applications. Here we will look at a few graphs that represent some examples of real-life application of quadratic functions.

Example 4 Area

Andrew has 100 feet of fence to enclose a rectangular tomato patch. He wants to find the dimensions of the rectangle that encloses most area.

Solution

We can find an equation for the area of the rectangle by looking at a sketch of the situation.

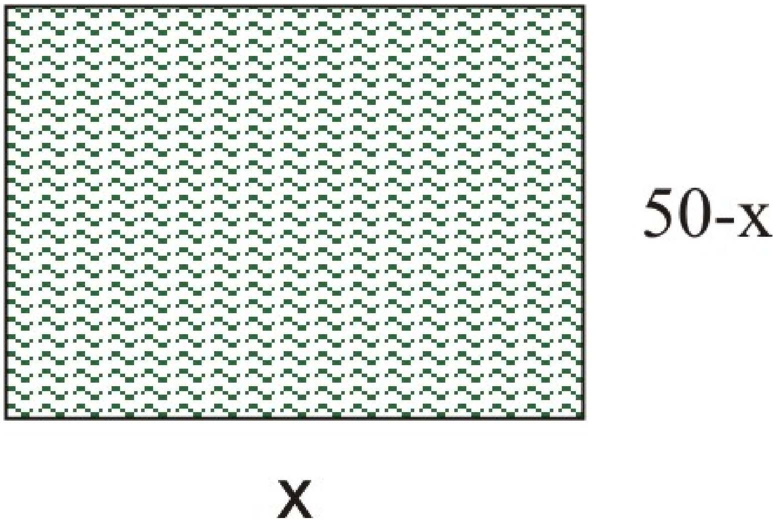


FIGURE 3.2

Let x be the length of the rectangle.

$50-x$ is the width of the rectangle (Remember there are two widths so its not $100-x$).

Let y be the area of the rectangle.

$$ath = \text{Area} = \text{length} \times \text{width} \Rightarrow y = x(50-x)$$

The following graph shows how the area of the rectangle depends on the length of the rectangle

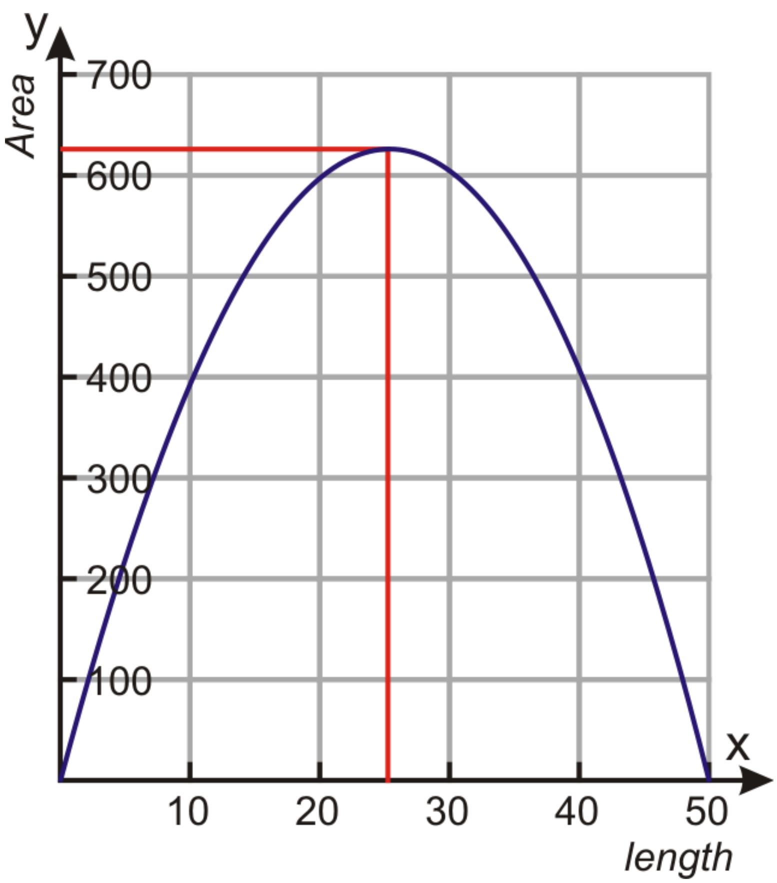


FIGURE 3.3

We can see from the graph that the highest value of the area occurs when the length of the rectangle is 25. The area of the rectangle for this side length equals 625. Notice that the width is also 25, which makes the shape a square with side length 25.

This is an example of an optimization problem.

Example 5 Projectile motion

Anne is playing golf. On the 4^{th} *tee, she hits a slow shot down the level fairway. The ball follows a parabolic path described by the equation,* $y = x - 0.04x^2$*. This relates the height of the ball y to the horizontal distance as the ball travels down the fairway. The distances are measured in feet. How far from the tee does the ball hit the ground? At what distance, x from the tee, does the ball attain its maximum height? What is the maximum height?*

Solution

Let's graph the equation of the path of the ball: $y = x - 0.04x^2$.

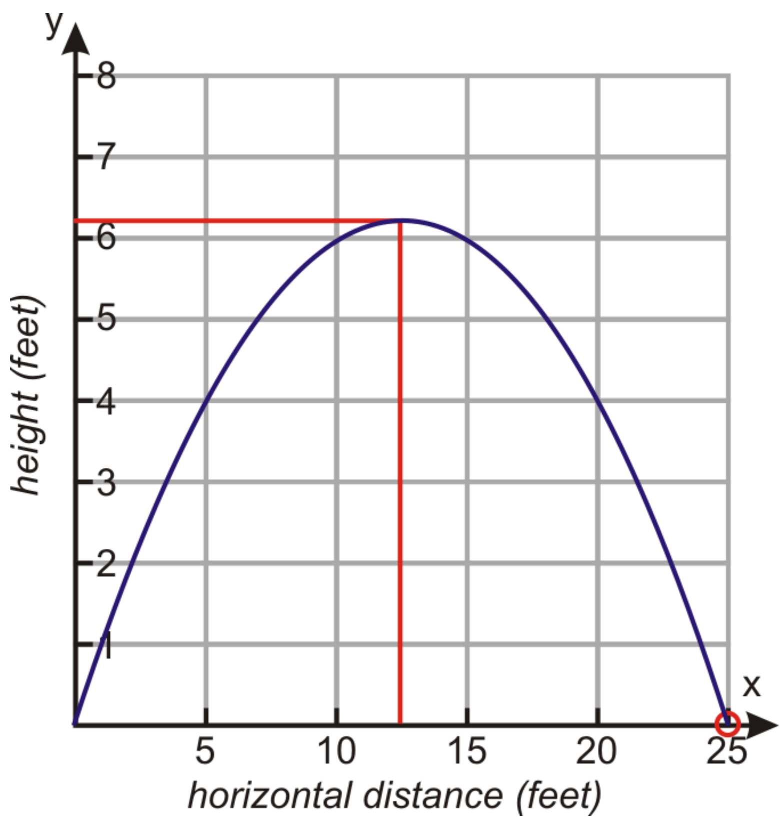


FIGURE 3.4

$y = x(1 - 0.04x)$ has solutions of $x = 0$ and $x = 25$

From the graph, we see that the ball hits the ground 25 feet from the tee.

We see that the maximum height is attained at 12.5 feet from the tee and the maximum height the ball reaches is 6.25 feet.

Review Questions

Rewrite the following functions in intercept form. Find the $x-$intercepts and the vertex.

1. $f(x) = x^2 - 2x - 8$
2. $f(x) = -x^2 + 10x - 21$
3. $f(x) = 2x^2 + 6x + 4$

Does the graph of the parabola turn up or down?

4. $f(x) = -2x^2 - 2x - 3$
5. $f(x) = 3x^2$
6. $f(x) = 16 - 4x^2$

The vertex of which parabola is higher?

7. $f(x) = x^2$ or $f(x) = 4x^2$
8. $f(x) = -2x^2$ or $f(x) = -2x^2 - 2$
9. $f(x) = 3x^2 - 3$ or $f(x) = 3x^2 - 6$

Which parabola is wider?

10. $f(x) = x^2$ or $f(x) = 4x^2$
11. $f(x) = 2x^2 + 4$ or $f(x) = \frac{1}{2}x^2 + 4$
12. $f(x) = -2x^2 - 2$ or $f(x) = -x^2 - 2$

Graph the following functions by making a table of values. Use the vertex and $x-$intercepts to help you pick values for the table.

13. $f(x) = 4x^2 - 4$
14. $f(x) = -x^2 + x + 12$
15. $f(x) = 2x^2 + 10x + 8$
16. $f(x) = \frac{1}{2}x^2 - 2x$
17. $f(x) = x - 2x^2$
18. $f(x) = 4x^2 - 8x + 4$

19. Nadia is throwing a ball to Peter. Peter does not catch the ball and it hits the ground. The graph shows the path of the ball as it flies through the air. The equation that describes the path of the ball is $y = 4 + 2x - 0.16x^2$. Here y is the height of the ball and x is the horizontal distance from Nadia. Both distances are measured in feet. How far from Nadia does the ball hit the ground? At what distance, x from Nadia, does the ball attain its maximum height? What is the maximum height?

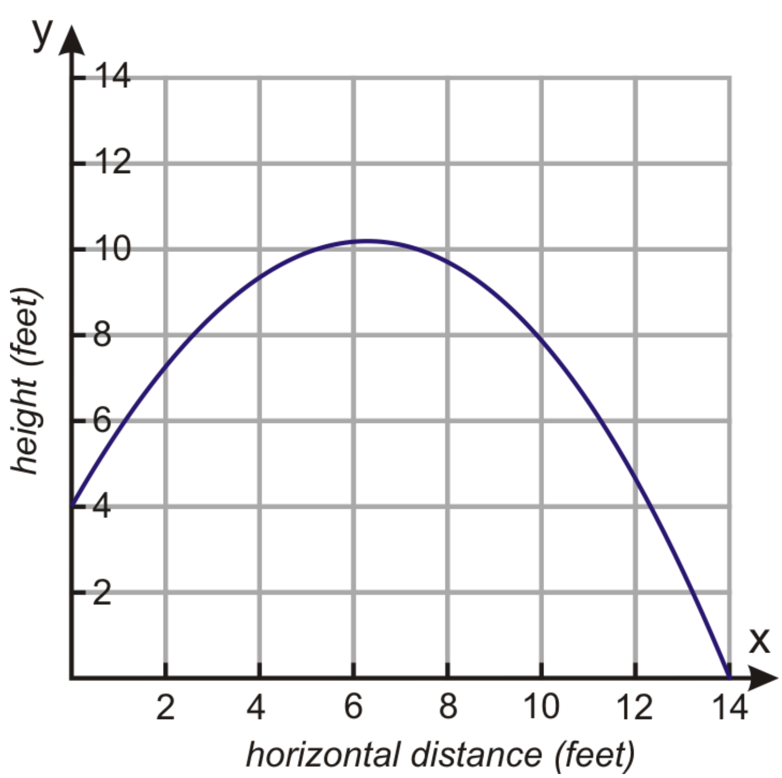


FIGURE 3.5

20. Peter wants to enclose a vegetable patch with 120 feet of fencing. He wants to put the vegetable against an existing wall, so he only needs fence for three of the sides. The equation for the area is given by $a = 120x - x^2$. From the graph find what dimensions of the rectangle would give him the greatest area.

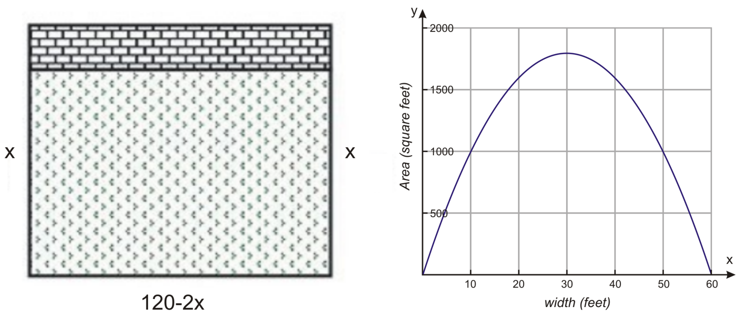


FIGURE 3.6

Review Answers

1. $x = -2, x = 4$ Vertex (1, -9)
2. $x = 3, x = 7$ Vertex (5, 4)
3. $x = -2, x = -1$ Vertex (-3.5, 7.5)
4. Down
5. Up
6. Down
7. $f(x) = x^2 + 4$
8. $f(x) = -2x^2$
9. $f(x) = 3x^2 - 3$
10. $f(x) = x^2$
11. $f(x) = \left(\frac{1}{2}\right)x^2 + 4$
12. $f(x) = -x^2 - 2$
13.

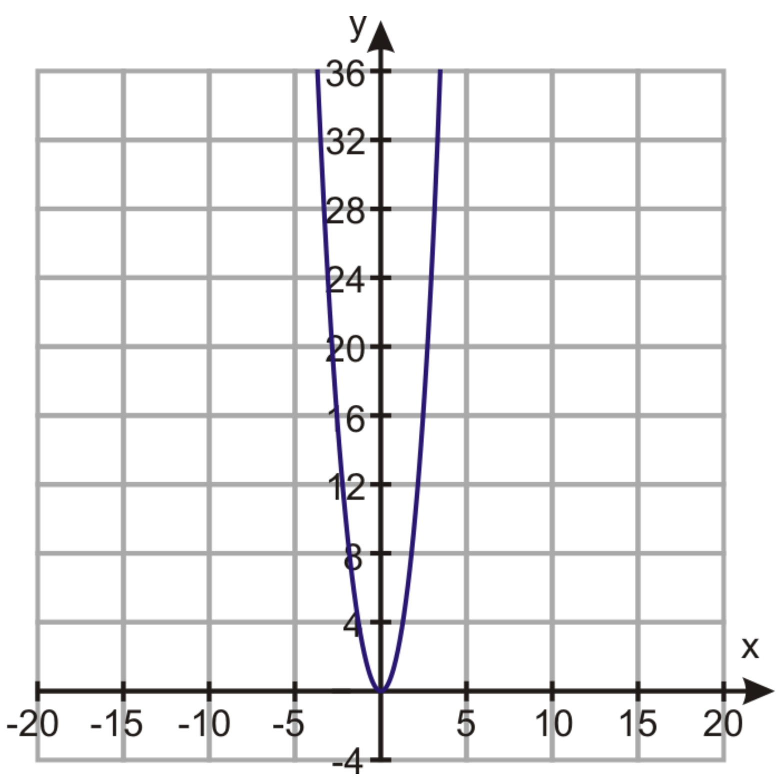


FIGURE 3.7

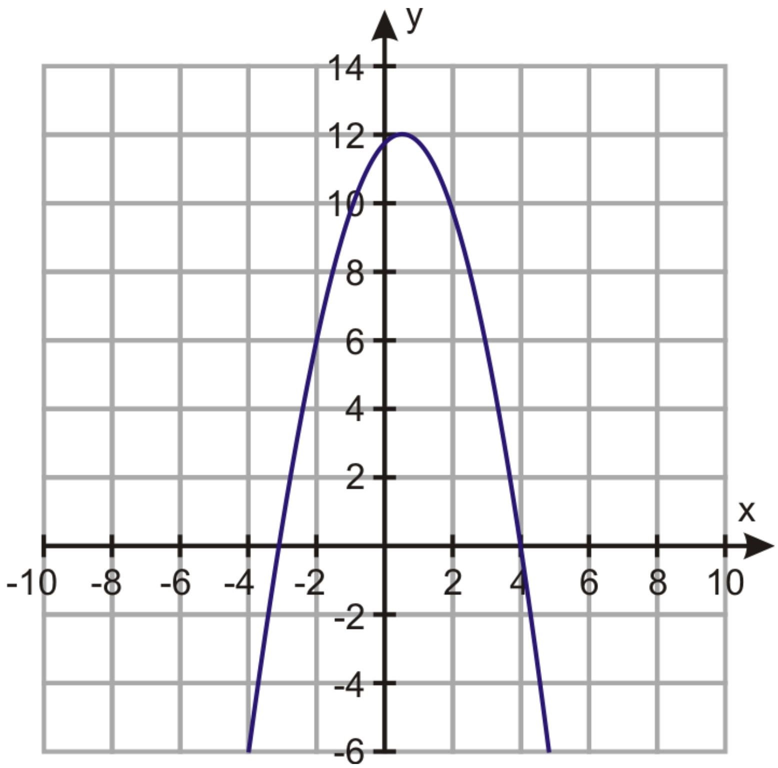
14. 

FIGURE 3.8

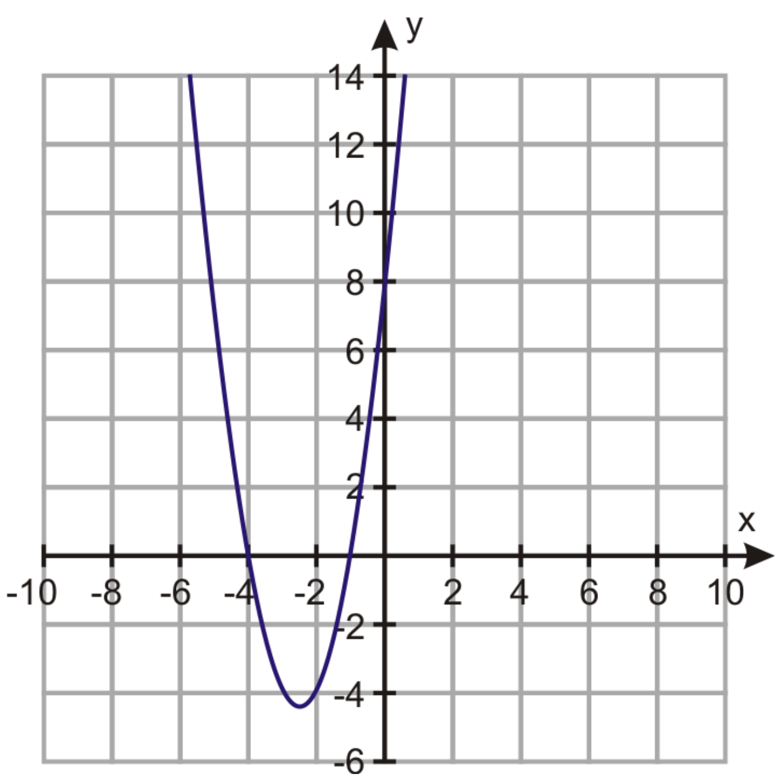
15. 

FIGURE 3.9

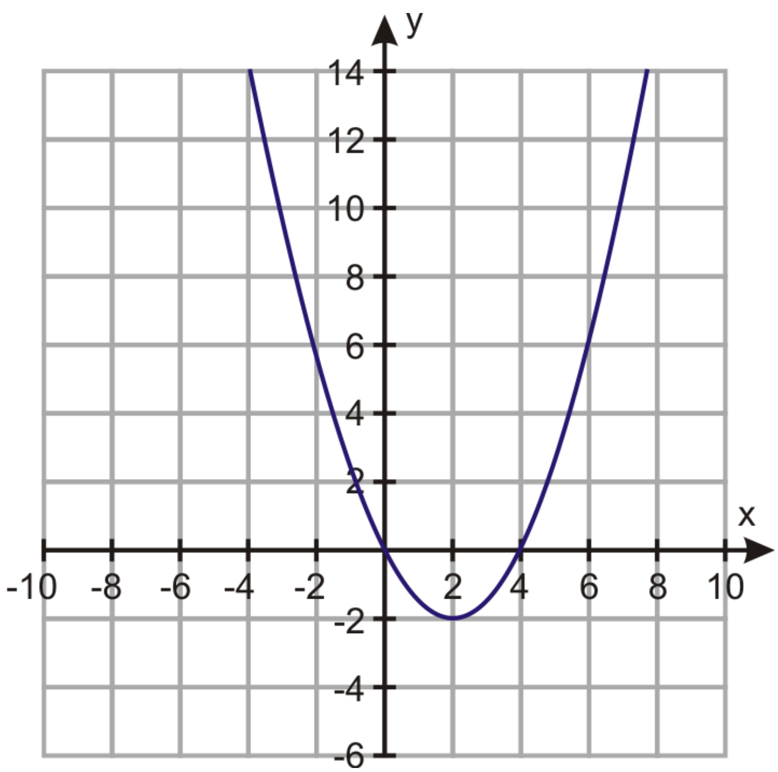
16. 

FIGURE 3.10

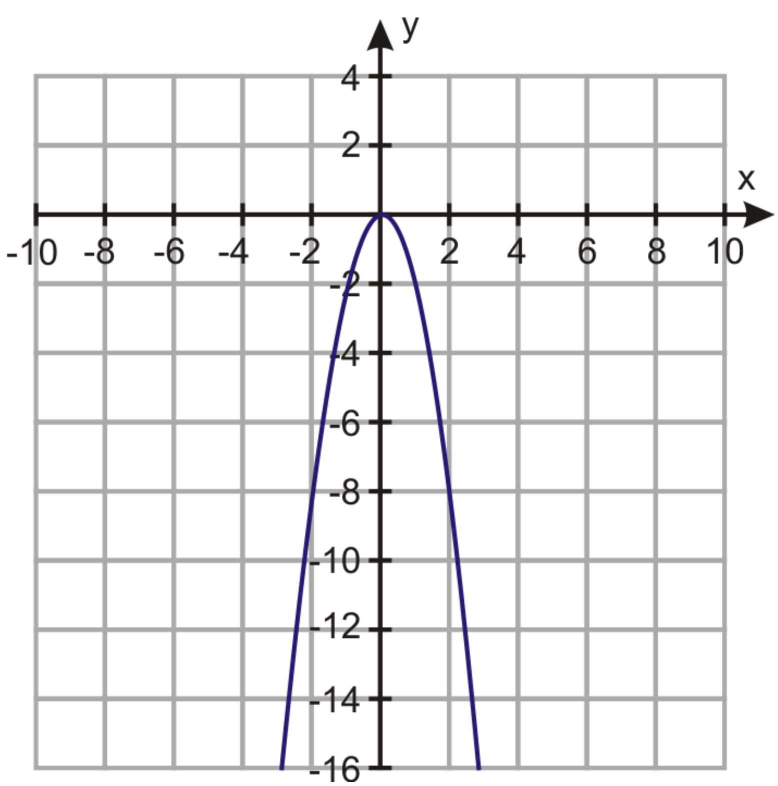
17. 

FIGURE 3.11

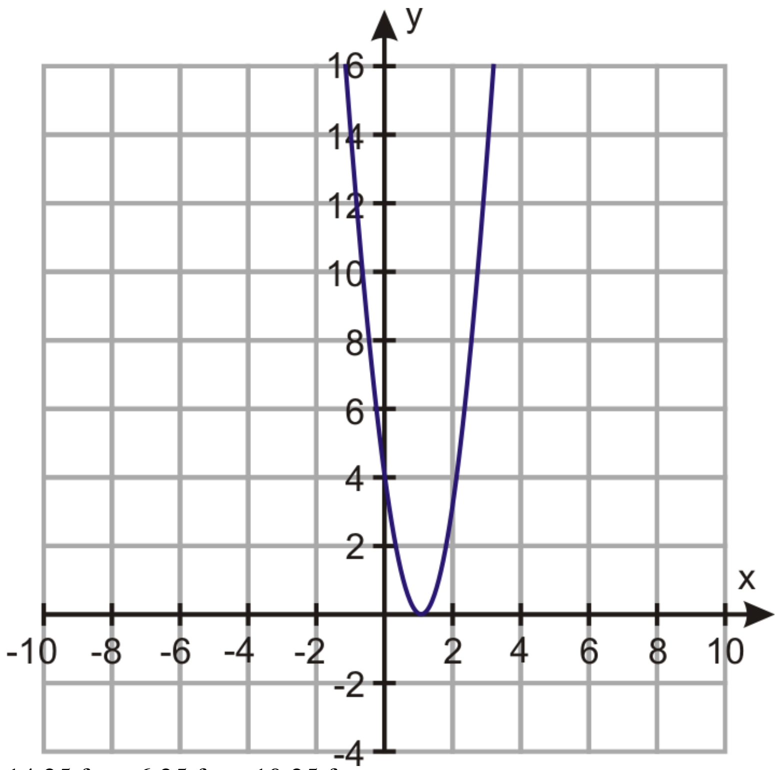
18. 

FIGURE 3.12

19. 14.25 feet, 6.25 feet, 10.25 feet
20. width $= 30$ $feet$, length $= 60$ $feet$

3.2 Quadratic Equations by Graphing

Learning Objectives

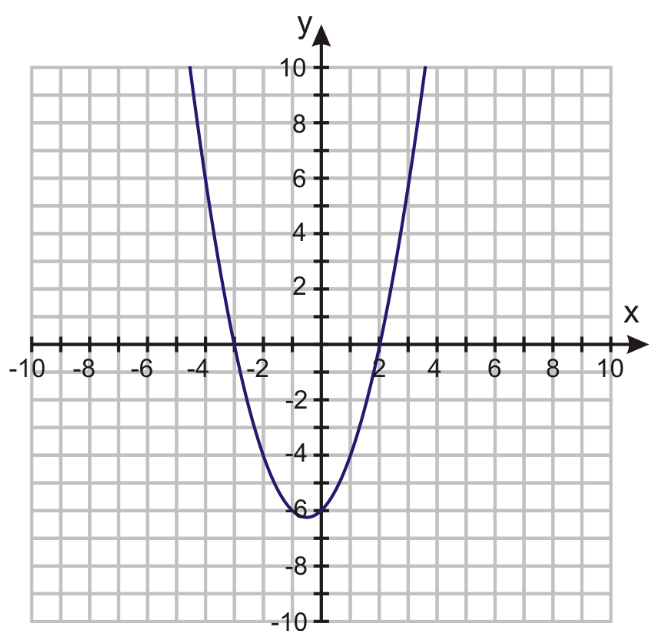
- Identify the number of solutions of quadratic equations.
- Solve quadratic equations by graphing.
- Find or approximate zeros of quadratic functions.
- Analyze quadratic functions using a graphing calculators.
- Solve real-world problems by graphing quadratic functions.

Introduction

In the last, section you learned how to graph quadratic equations. You saw that finding the $x-$intercepts of a parabola is important because it tells us where the graph crosses the $x-$axis. and it also lets us find the vertex of the parabola. When we are asked to find the **solutions** of the quadratic equation in the form $ax^2+bx+c=0$, we are basically asked to find the $x-$intercepts of the quadratic function.

Finding the $x-$intercepts of a parabola is also called finding the **roots** or **zeros** of the function.

Identify the Number of Solutions of Quadratic Equations



The graph of a quadratic equation is very useful in helping us identify how many solutions and what types of solutions a function has. There are three different situations that occur when graphing a quadratic function.

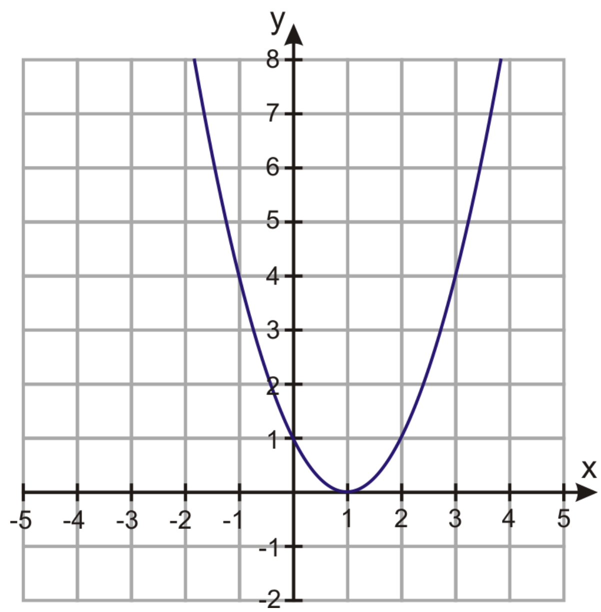
Case 1 The parabola crosses the $x-$axis at two points.

An example of this is $y=x^2+x-6$.

We can find the solutions to equation $x^2+x-6=0$ by setting $y=0$. We solve the equation by factoring $(x+3)(x-2)=0$ so $x=-3$ or $x=2$.

Another way to find the solutions is to graph the function and read the $x-$intercepts from the graph. We see that the parabola crosses the $x-$axis at $x = -3$ and $x = 2$.

When the graph of a quadratic function crosses the $x-$axis at two points, we get **two distinct solutions** to the quadratic equation.



Case 2 The parabola touches the $x-$axis at one point.

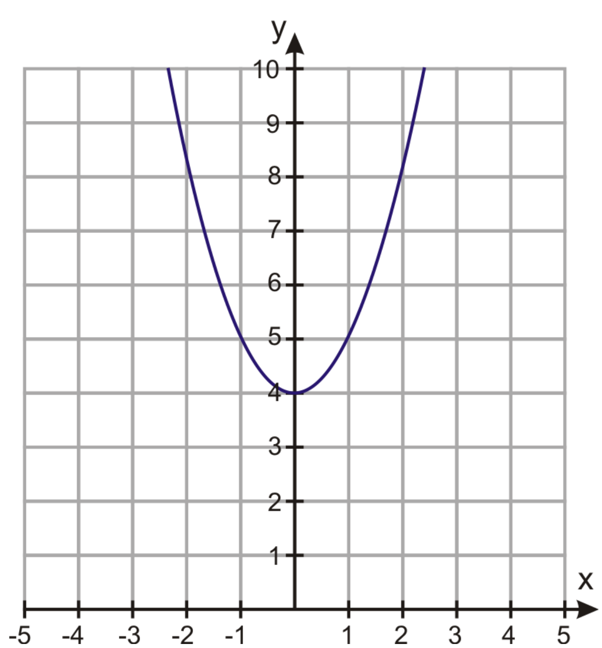
An example of this is $y = x^2 - 2x + 1$.

We can also solve this equation by factoring. If we set $y = 0$ and factor, we obtain $(x-1)^2$ so $x = 1$.

Since the quadratic function is a perfect square, we obtain only one solution for the equation.

Here is what the graph of this function looks like. We see that the graph touches the $x-$axis at point $x = 1$.

When the graph of a quadratic function touches the $x-$axis at one point, the quadratic equation has one solution and the solution is called a **double root.**



Case 3 The parabola does not cross or touch the $x-$axis.

An example of this is $y = x^2 + 4$. If we set $y = 0$ we get $x^2 + 4 = 0$. This quadratic polynomial does not factor and the equation $x^2 = -4$ has no real solutions. When we look at the graph of this function, we see that the parabola does not cross or touch the $x-$axis.

When the graph of a quadratic function does not cross or touch the $x-$axis, the quadratic equation has **no real solutions.**

Solve Quadratic Equations by Graphing.

So far we have found the solutions to graphing equations using factoring. However, there are very few functions in real life that factor easily. As you just saw, graphing the function gives a lot of information about the solutions. We can find exact or approximate solutions to quadratic equations by graphing the function associated with it.

Example 1

Find the solutions to the following quadratic equations by graphing.

a) $-x^2 + 3 = 0$

b) $2x^2 + 5x - 7 = 0$

c) $-x^2 + x - 3 = 0$

Solution

Let's graph each equation. Unfortunately none of these functions can be rewritten in intercept form because we cannot factor the right hand side. This means that you cannot find the $x-$intercept and vertex before graphing since you have not learned methods other than factoring to do that.

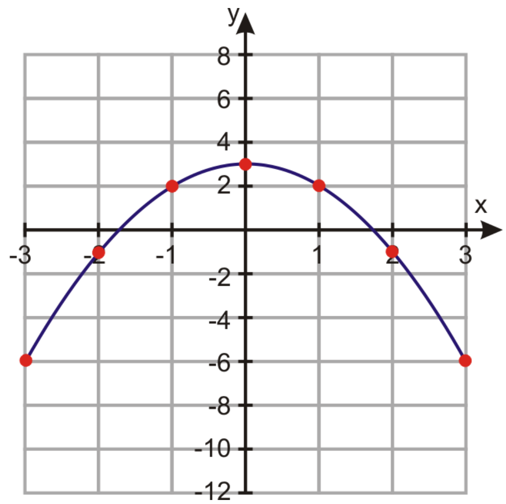
a) To find the solution to $-x^2 + 3 = 0$, we need to find the $x-$intercepts of $y = -x^2 + 3$.

Let's make a table of values so we can graph the function.

TABLE 3.9:

x	$y = -x^2 + 3$
-3	$y = -(-3)^2 + 3 = -6$
-2	$y = -(-2)^2 + 3 = -1$
-1	$y = -(-1)^2 + 3 = 2$
0	$y = -(0)^2 + 3 = 3$
1	$y = -(-1)^2 + 3 = 2$
2	$y = -(2)^2 + 3 = -1$
3	$y = -(3)^2 + 3 = -6$

We plot the points and get the following graph:



From the graph we can read that the $x-$intercepts are approximately $x = 1.7$**and** $x = -1.7$.

These are the solutions to the equation $-x^2 + 3 = 0$.

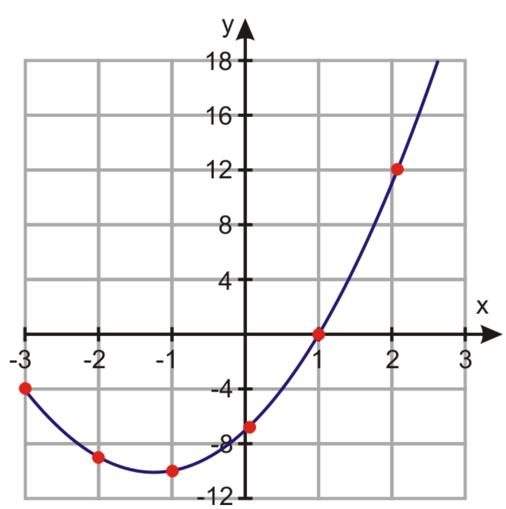
b) To solve the equation $2x^2 + 5x - 7 = 0$ we need to find the $x-$intercepts of $y = 2x^2 + 5 - 7$.

Let's make a table of values so we can graph the function.

TABLE 3.10:

x	$y = 2x^2 + 5x - 7$
-3	$y = 2(-3)^2 + 5(-3) - 7 = -4$
-2	$y = 2(-2)^2 + 5(-2) - 7 = -9$
-1	$y = 2(-1)^2 + 5(-1) - 7 = -10$
0	$y = 2(0)^0 + 5(0) - 7 = -7$
1	$y = 2(1)^2 + 5(1) - 7 = 0$
2	$y = 2(2)^2 + 5(2) - 7 = 11$
3	$y = 2(3)^2 + 5(3) - 7 = 26$

We plot the points and get the following graph:

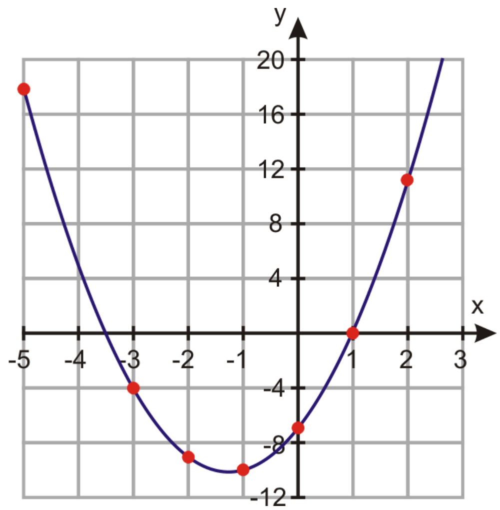


Since we can only see one $x-$intercept on this graph, we need to pick more points smaller than $x = -3$ and re-draw the graph.

TABLE 3.11:

x	$y = 2x^2 + 5x - 7$
-5	$y = 2(-5)^2 + 5(-5) - 7 = 18$
-4	$y = 2(-4)^2 + 5(-4) - 7 = 5$

Here is the graph again with both $x-$intercepts showing:



From the graph we can read that the $x-$intercepts are $x = 1$**and** $x = -3.5$.

These are the solutions to equation $2x^2 + 5x - 7 = 0$.

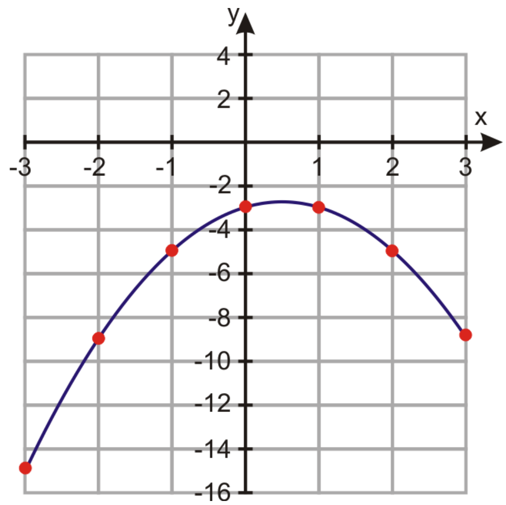
c) To solve the equation $-x^2 + x - 3 = 0$ we need to find the $x-$intercepts of $y = -x^2 + x - 3$.

Let's make a table of values so we can graph the function.

TABLE 3.12:

x	$y = -x^2 + x - 3$
-3	$y = -(-3)^2 + (-3) - 3 = -15$
-2	$y = -(-2)^2 + (-2) - 3 = -9$
-1	$y = -(-1)^2 + (-1) - 3 = -5$
0	$y = -(0)^2 + (0) - 3 = -3$
1	$y = -(1)^2 + (1) - 3 = -3$
2	$y = -(-2)^2 + (2) - 3 = -5$
3	$y = -(3)^2 + (3) - 3 = -9$

We plot the points and get the following graph:



This graph has no $x-$intercepts, so the equation $-x^2+x-3=0$ has **no real solutions.**

Find or Approximate Zeros of Quadratic Functions

From the graph of a quadratic function $y=ax^2+bx+c$, we can find the **roots** or **zeros** of the function. The zeros are also the $x-$intercepts of the graph, and they solve the equation $ax^2+bx+c=0$. When the zeros of the function are integer values, it is easy to obtain exact values from reading the graph. When the zeros are not integers we must approximate their value.

Let's do more examples of finding zeros of quadratic functions.

Example 2 Find the zeros of the following quadratic functions.

a) $y=-x^2+4x-4$

b) $y=3x^2-5x$

Solution

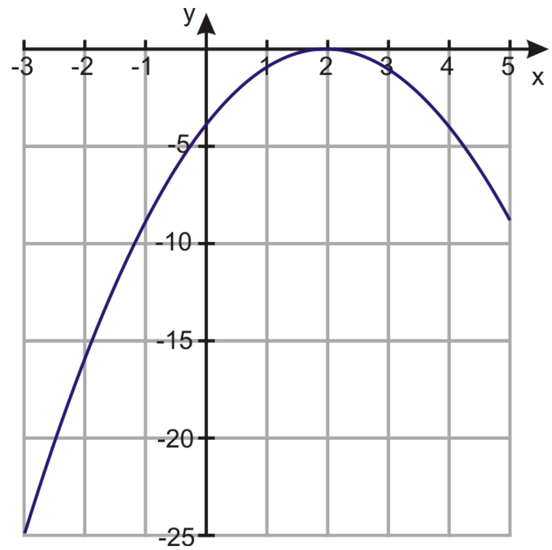
a) Graph the function $y=-x^2+4x-4$ and read the values of the $x-$intercepts from the graph.

Let's make a table of values.

TABLE 3.13:

x	$y=-x^2+4x-4$
-3	$y=-(-3)^2+4(-3)-4=-25$
-2	$y=-(-2)^2+4(-2)-4=-16$
-1	$y=-(-1)^2+4(-1)-4=-9$
0	$y=-(0)^2+4(0)-4=-4$
1	$y=-(1)^2+4(1)-4=-1$
2	$y=-(2)^2+4(2)-4=0$
3	$y=-(3)^2+4(3)-4=-1$
4	$y=-(4)^2+4(4)-4=-4$
5	$y=-(5)^2+4(5)-4=-9$

Here is the graph of this function.



The function has a **double root** at $x = 2$.

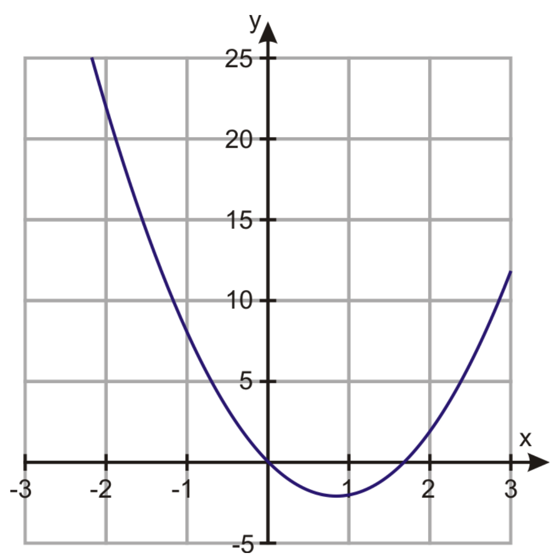
b) Graph the function $y = 3x^2 - 5x$ and read the $x-$intercepts from the graph.

Let's make a table of values.

TABLE 3.14:

x	$y = 3x^2 - 5x$
-3	$y = 3(-3)^2 - 5(-3) = 42$
-2	$y = 3(-2)^2 - 5(-2) = 22$
-1	$y = 3(-1)^2 - 5(-1) = 8$
0	$y = 3(0)^2 - 5(0) = 0$
1	$y = 3(1)^2 - 5(1) = -2$
2	$y = 3(2)^2 - 5(2) = 2$
3	$y = 3(3)^2 - 5(3) = 12$

Here is the graph of this function.



The function has two roots: $x = 0$**and** $x \approx 1.7$.

Analyze Quadratic Functions Using a Graphing Calculator

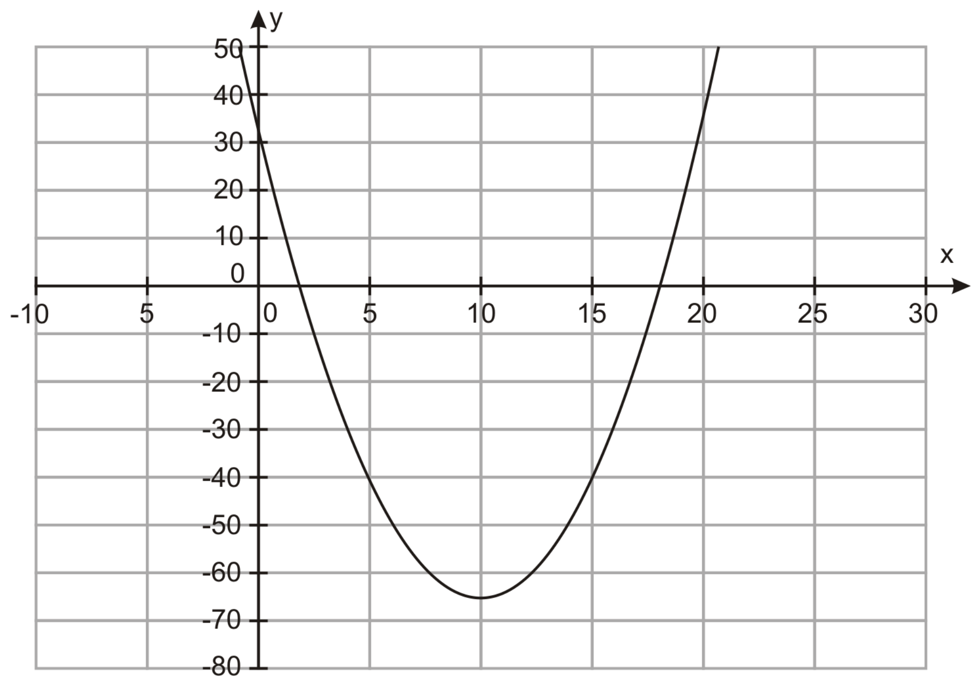
A graphing calculator is very useful for graphing quadratic functions. Once the function is graphed, we can use the calculator to find important information such as the roots of the function or the vertex of the function.

Example 3

Let's use the graphing calculator to analyze the graph of $y = x^2 - 20x + 35$.

1. **Graph** the function.

Press the **[Y=]** button and enter "$X^2 - 20X + 35$" next to **[**$Y_1 =$**]**.(Note, X is one of the buttons on the calculator)



Press the **[GRAPH]** button. This is the plot you should see. If this is not what you see change the window size. For the graph to the right, we used window size of XMIN $= -10$, XMAX $= 30$ and YMIN $= -80$, YMAX $= 50$. To change window size, press the **[WINDOW]** button.

2. Find the **roots.**

There are at least three ways to find the roots

Use **[TRACE]** to scroll over the $x-$intercepts. The approximate value of the roots will be shown on the screen.You can improve your estimate by zooming in.

OR

Use **[TABLE]** and scroll through the values until you find values of Y equal to zero. You can change the accuracy of the solution by setting the step size with the **[TBLSET]** function.

OR

Use **[2nd] [TRACE]** (i.e. 'calc' button) and use option 'zero'.

Move cursor to the left of one of the roots and press **[ENTER]**.

Move cursor to the right of the same root and press **[ENTER]**.

Move cursor close to the root and press **[ENTER]**.

The screen will show the value of the root. For the left side root, we obtained $x = 1.9$.

Repeat the procedure for the other root. For the right side root, we obtained $x = 18$.

3. Find the **vertex**

There are three ways to find the vertex.

Use **[TRACE]** to scroll over the highest or lowest point on the graph. The approximate value of the roots will be shown on the screen.

OR

Use **[TABLE]** and scroll through the values until you find values the lowest or highest values of Y.

You can change the accuracy of the solution by setting the step size with the **[TBLSET]** function.

OR

Use **[2nd] [TRACE]** and use option 'maximum' if the vertex is a maximum or option 'minimum' if the vertex is a minimum.

Move cursor to the left of the vertex and press **[ENTER]**.

Move cursor to the right of the vertex and press **[ENTER]**.

Move cursor close to the vertex and press **[ENTER]**.

The screen will show the x and y values of the vertex.

For this example, we obtained $x = 10$ and $x = -65$.

Solve Real-World Problems by Graphing Quadratic Functions

We will now use the methods we learned so far to solve some examples of real-world problems using quadratic functions.

Example 4 Projectile motion

Andrew is an avid archer. He launches an arrow that takes a parabolic path. Here is the equation of the height of the ball with respect to time.

$$y = -4.9t^2 + 48t$$

Here y is the height in meters and t is the time in seconds. Find how long it takes the arrow to come back to the ground.

Solution

Let's graph the equation by making a table of values.

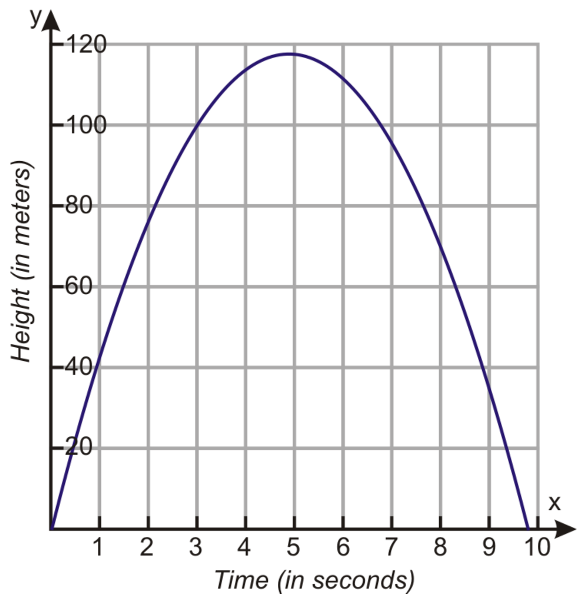
TABLE 3.15:

t	$y = -4.9t^2 + 48t$
0	$y = -4.9(0)^2 + 48(0) = 0$
1	$y = -4.9(1)^2 + 48(1) = 43.1$
2	$y = -4.9(2)^2 + 48(2) = 76.4$
3	$y = -4.9(3)^2 + 48(3) = 99.9$
4	$y = -4.9(4)^2 + 48(4) = 113.6$
5	$y = -4.9(5)^2 + 48(5) = 117.5$
6	$y = -4.9(6)^2 + 48(6) = 111.6$
7	$y = -4.9(7)^2 + 48(7) = 95.9$

TABLE 3.15: (continued)

t	$y = -4.9t^2 + 48t$
8	$y = -4.9(8)^2 + 48(8) = 70.4$
9	$y = -4.9(9)^2 + 48(9) = 35.1$
10	$y = -4.9(10)^2 + 48(10) = -10$

Here is the graph of the function.



The roots of the function are approximately $x = 0$ *sec* and $x = 9.8$ *sec*. The first root says that at time 0 seconds the height of the arrow is 0 meters. The second root says that it takes approximately 9.8 seconds for the arrow to return back to the ground.

Review Questions

Find the solutions of the following equations by graphing.

1. $x^2 + 3x + 6 = 0$
2. $-2x^2 + x + 4 = 0$
3. $x^2 - 9 = 0$
4. $x^2 + 6x + 9 = 0$
5. $10x^2 - 3x^2 = 0$
6. $\frac{1}{2}x^2 - 2x + 3 = 0$

Find the roots of the following quadratic functions by graphing.

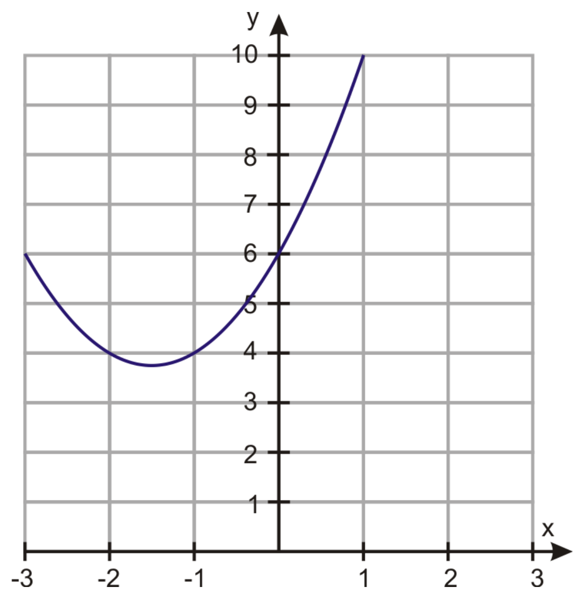
1. $y = -3x^2 + 4x - 1$
2. $y = 9 - 4x^2$
3. $y = x^2 + 7x + 2$
4. $y = -x^2 - 10x - 25$
5. $y = 2x^2 - 3x$
6. $y = x^2 - 2x + 5$Using your graphing calculator
   (a) Find the roots of the quadratic polynomials.

(b) Find the vertex of the quadratic polynomials.

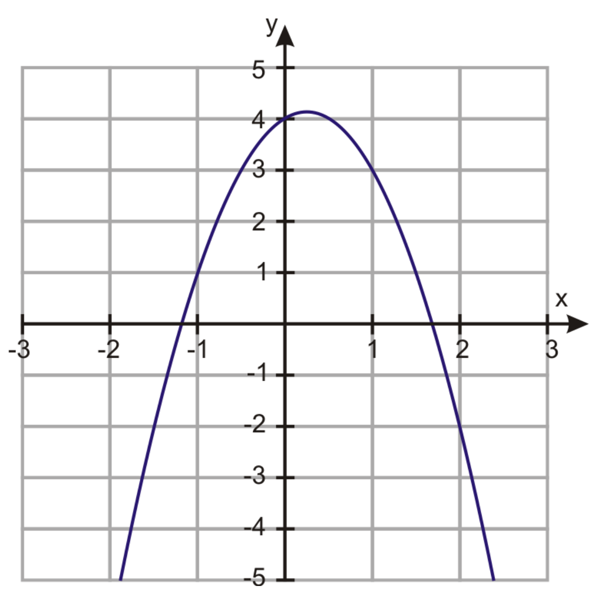
1. $y = x^2 + 12x + 5$
2. $y = x^2 + 3x + 6$
3. $y = -x^2 - 3x + 9$
4. Peter throws a ball and it takes a parabolic path. Here is the equation of the height of the ball with respect to time:
   $y = -16t^2 + 60t$
   Here y is the height in feet and t is the time in seconds. Find how long it takes the ball to come back to the ground.
5. Use your graphing calculator to solve Ex. 5. You should get the same answers as we did graphing by hand but a lot quicker!

Review Answers

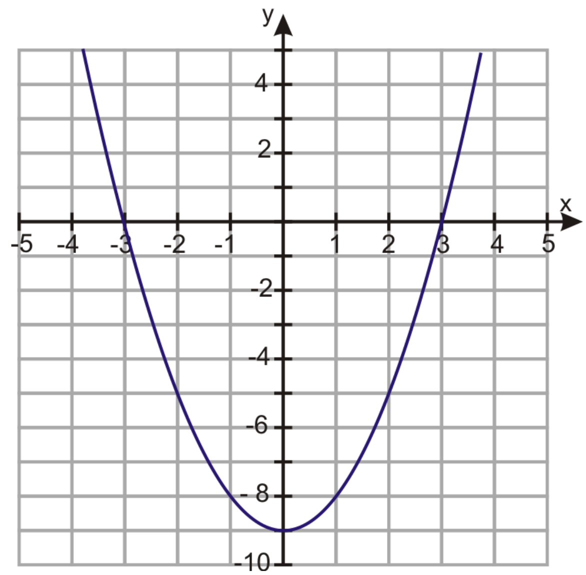
1. No real solutions



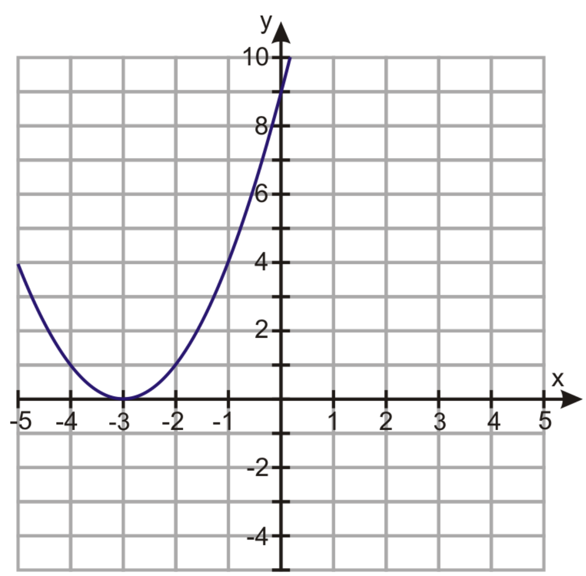
2. $x = -1.2, x = 1.87$



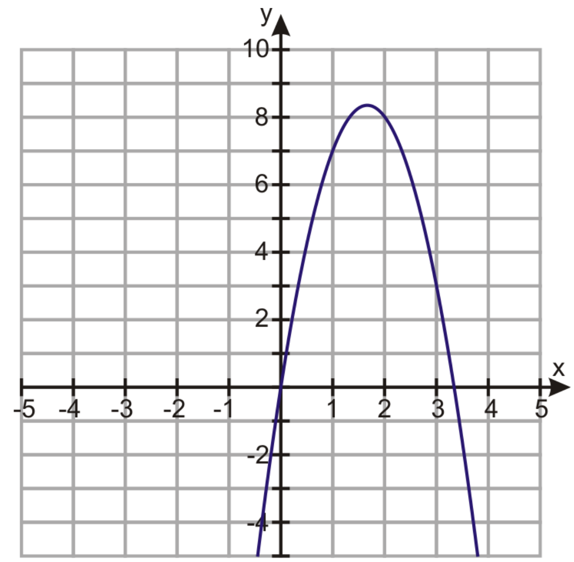
3. $x = -3, x = 3$



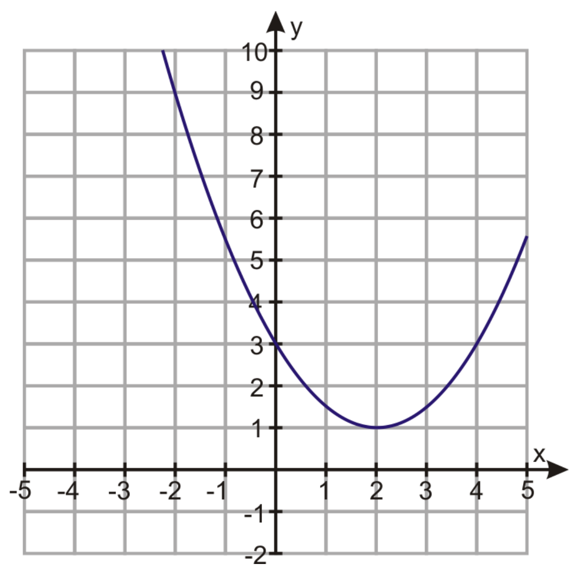
4. $x = -3$double root



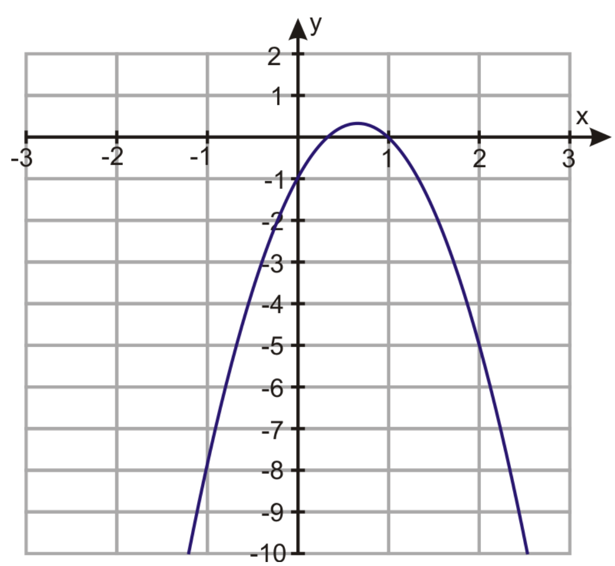
5. $x = 0, x = 3.23$



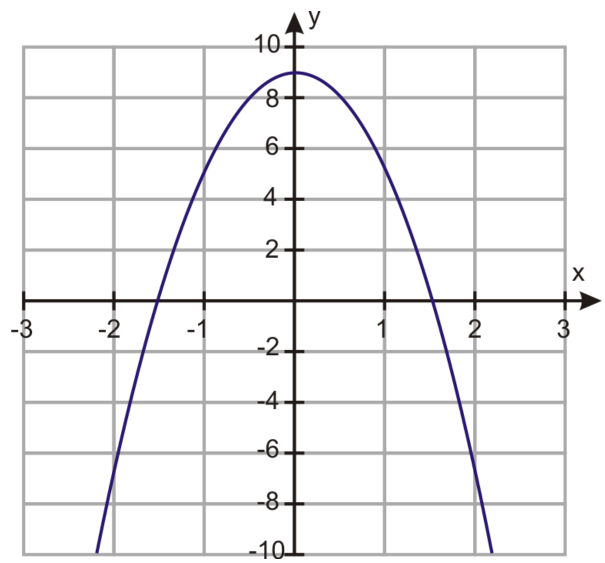
6. No real solutions.



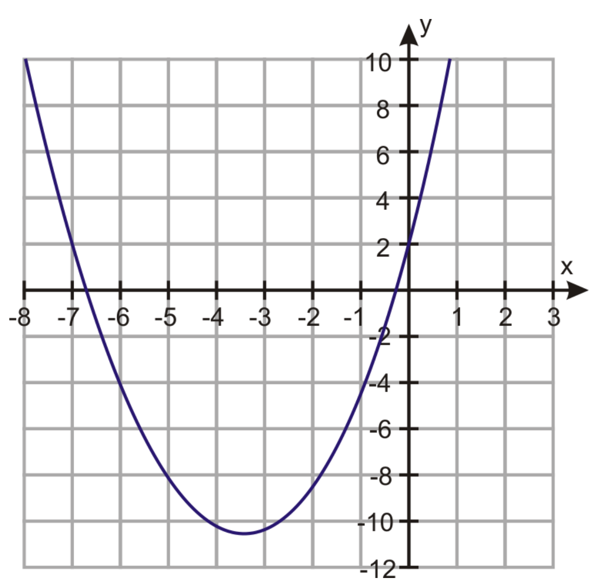
7. $x = 0.3, x = 1$



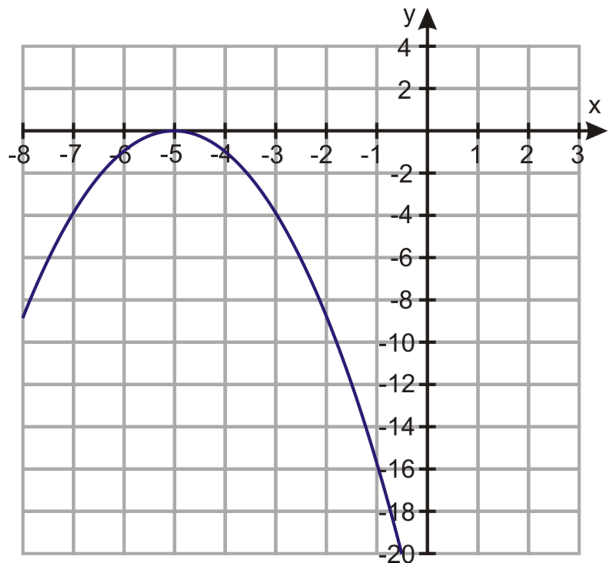
8. $x = -1.5, x = 1.5$



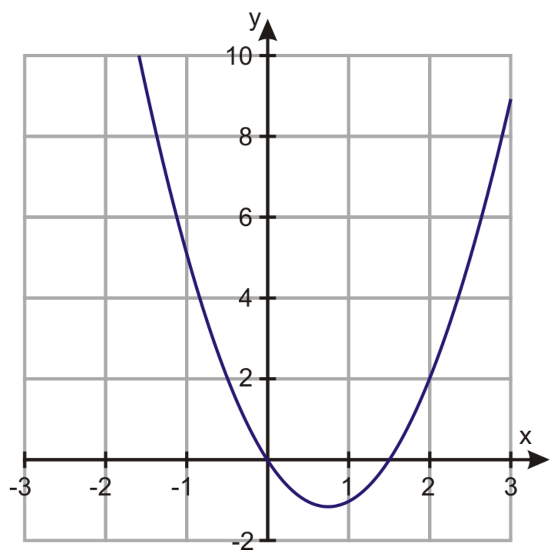
9. $x = -6.7, x = 0.3$



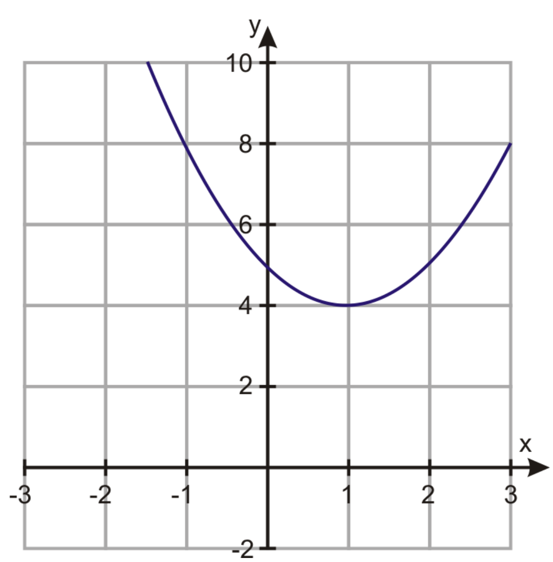
10. $x = -5$double root



11. $x = 0, x = 1.5$



12. No real solutions.



13. x ≈ -11.6 and x ≈ -0.4. Vertex (-6, -31)
14. No real solution. Vertex (-3/2, 15/4)
15. time = 3.75 *second*

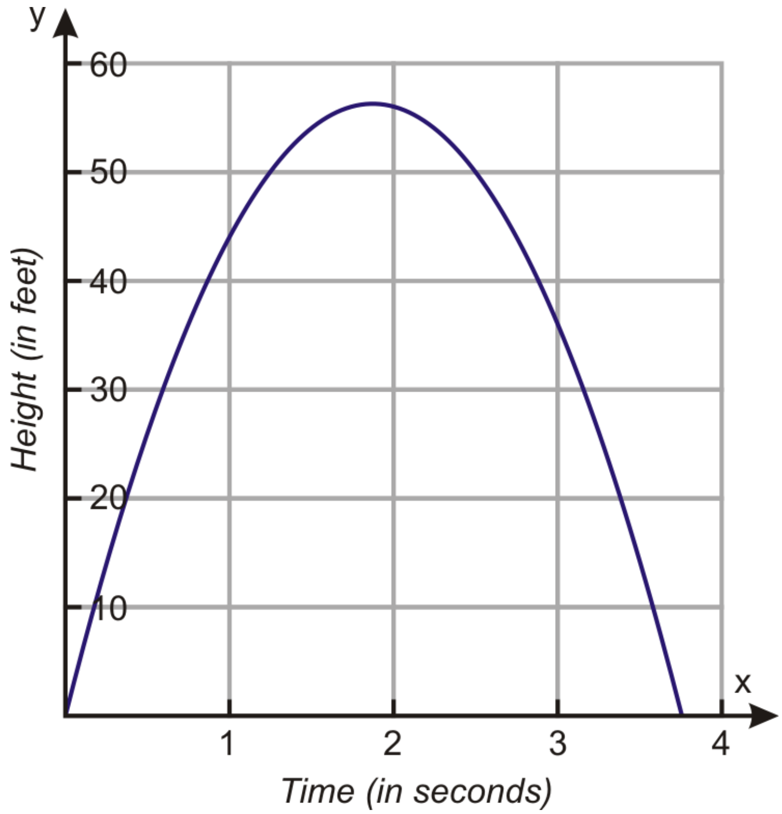


FIGURE 3.13

3.3 Quadratic Equations by Square Roots

Learning objectives

- Solve quadratic equations involving perfect squares.
- Approximate solutions of quadratic equations.
- Solve real-world problems using quadratic functions and square roots.

Introduction

So far you know how to solve quadratic equations by factoring. However, this method works only if a quadratic polynomial can be factored. Unfortunately, in practice, most quadratic polynomials are not factorable. In this section you will continue learning new methods that can be used in solving quadratic equations. In particular, we will examine equations in which we can take the square root of both sides of the equation in order to arrive at the result.

Solve Quadratic Equations Involving Perfect Squares

Let's first examine quadratic equation of the type

$$x^2 - c = 0$$

We can solve this equation by isolating the x^2 term: $x^2 = c$

Once the x^2 term is isolated we can take the square root of both sides of the equation. Remember that when we take the square root we get two answers: the positive square root and the negative square root:

$$x = \sqrt{c} \qquad \text{and} \qquad x = -\sqrt{c}$$

Often this is written as $x = \pm\sqrt{c}$.

Example 1

Solve the following quadratic equations

a) $x^2 - 4 = 0$

b) $x^2 - 25 = 0$

Solution

a) $x^2 - 4 = 0$

Isolate the x^2 $x^2 = 4$

Take the square root of both sides $x = \sqrt{4}$ and $x = -\sqrt{4}$

Answer $x = 2$ and $x = -2$

b) $x^2 - 25 = 0$

Isolate the x^2	$x^2 = 25$
Take the square root of both sides	$x = \sqrt{25}$ and $x = -\sqrt{25}$

Answer $x = 5$ and $x = -5$

Another type of equation where we can find the solution using the square root is

$$ax^2 - c = 0$$

We can solve this equation by isolating the x^2 term

$$ax^2 = c$$
$$x^2 = \frac{c}{a}$$

Now we can take the square root of both sides of the equation.

$$x = \sqrt{\frac{c}{a}} \qquad \text{and} \qquad x = -\sqrt{\frac{c}{a}}$$

Often this is written as $x = \pm\sqrt{\frac{c}{a}}$.

Example 2

Solve the following quadratic equations.

a) $9x^2 - 16 = 0$

b) $81x^2 - 1 = 0$

Solution

a) $9x^2 - 16 = 0$

Isolate the x^2.	$9x^2 = 16$
	$x^2 = \frac{16}{9}$

Take the square root of both sides. $x = \sqrt{\frac{16}{9}}$ and $x = -\sqrt{169}$

Answer: $x = \frac{4}{3}$ and $x = -\frac{4}{3}$

b) $81x^2 - 1 = 0$

Isolate the x^2	$81x^2 = 1$
	$x^2 = \frac{1}{81}$

Take the square root of both sides $x = \sqrt{\frac{1}{81}}$ and $x = -\sqrt{\frac{1}{81}}$

Answer $x = \frac{1}{9}$ and $x = -\frac{1}{9}$

As you have seen previously, some quadratic equations have no real solutions.

Example 3

Solve the following quadratic equations.

a) $x^2 + 1 = 0$

b) $4x^2 + 9 = 0$

Solution

a) $x^2 + 1 = 0$

Isolate the x^2 $\qquad x^2 = -1$

Take the square root of both sides: $x = \sqrt{-1}$ and $x = -\sqrt{-1}$

Answer Square roots of negative numbers do not give real number results, so there are **no real solutions** to this equation.

b) $4x^2 + 9 = 0$

Isolate the x^2

$$4x^2 = -9$$
$$x^2 = -\frac{9}{4}$$

Take the square root of both sides $x = \sqrt{-\frac{9}{4}}$ and $x = -\sqrt{-\frac{9}{4}}$

Answer There are **no real solutions.**

We can also use the square root function in some quadratic equations where one side of the equation is a perfect square. This is true if an equation is of the form

$$(x-2)^2 = 9$$

Both sides of the equation are perfect squares. We take the square root of both sides.

$x - 2 = 3$ and $x - 2 = -3$

Solve both equations

Answer $x = 5$ and $x = -1$

Example 4

Solve the following quadratic equations.

a) $(x-1)^2 = 4$

b) $(x+3)^2 = 1$

Solution

a) $(x-1)^2 = 4$

Take the square root of both sides.	$x-1=2$ and $x-1=-2$
Solve each equation.	$x=3$ and $x=-1$

Answer $x=3$ and $x=-1$

b) $(x+3)^2 = 1$

Take the square root of both sides.	$x+3=1$ and $x+3=-1$
Solve each equation.	$x=-2$ and $x=-4$

It might be necessary to factor the right hand side of the equation as a perfect square before applying the method outlined above.

Example 5

Solve the following quadratic equations.

a) $x^2+8x+16=25$

b) $4x^2-40x+25=9$

Solution

a) $x^2+8x+16=25$

Factor the right hand side.	$x^2+8x+16=(x+4)^2$	so $(x+4)^2=25$
Take the square root of both sides.	$x+4=5$ and $x+4=-5$	
Solve each equation.	$x=1$ and $x=-9$	

Answer $x=1$ and $x=-9$

b) $4x^2-20x+25=9$

Factor the right hand side.	$4x^2-20x+25=(2x-5)^2$	so $(2x-5)^2=9$
Take the square root of both sides.	$2x-5=3$ and $2x-5=-3$	
Solve each equation.	$2x=8$ and $2x=2$	

Answer $x=4$ and $x=1$

Approximate Solutions of Quadratic Equations

We use the methods we learned so far in this section to find approximate solutions to quadratic equations. We can get approximate solutions when taking the square root does not give an exact answer.

Example 6

Solve the following quadratic equations.

a) $x^2-3=0$

b) $2x^2 - 9 = 0$

Solution

a)

$$\text{Isolate the } x^2. \qquad x^2 = 3$$
$$\text{Take the square root of both sides.} \qquad x = \sqrt{3} \text{ and } x = -\sqrt{3}$$

Answer $x \approx 1.73$ and $x \approx -1.73$

b)

$$\text{Isolate the } x^2. \qquad 2x^2 = 9 \text{ so } x^2 = \frac{9}{2}$$
$$\text{Take the square root of both sides.} \qquad x = \sqrt{\frac{9}{2}} \text{ and } x = -\sqrt{\frac{9}{2}}$$

Answer $x \approx 2.12$ and $x \approx -2.12$

Example 7

Solve the following quadratic equations.

a) $(2x+5)^2 = 10$

b) $x^2 - 2x + 1 = 5$

Solution.

a)

$$\text{Take the square root of both sides.} \qquad 2x+5 = \sqrt{10} \text{ and } 2x+5 = -\sqrt{10}$$
$$\text{Solve both equations.} \qquad x = \frac{-5+\sqrt{10}}{2} \text{ and } x = \frac{-5-\sqrt{10}}{2}$$

Answer $x \approx -0.92$ and $x \approx -4.08$

b)

$$\text{Factor the right hand side.} \qquad (x-1)^2 = 5$$
$$\text{Take the square root of both sides.} \qquad x-1 = \sqrt{5} \text{ and } x-1 = -\sqrt{5}$$
$$\text{Solve each equation.} \qquad x = 1+\sqrt{5} \text{ and } x = 1-\sqrt{5}$$

Answer $x \approx 3.24$ and $x \approx -1.24$

Solve Real-World Problems Using Quadratic Functions and Square Roots

There are many real-world problems that require the use of quadratic equations in order to arrive at the solution. In this section, we will examine problems about objects falling under the influence of gravity. When objects are **dropped** from a height, they have no initial velocity and the force that makes them move towards the ground is due to gravity. The acceleration of gravity on earth is given by

$$g = -9.8\ m/s^2 \qquad \text{or} \qquad g = -32\ ft/s^2$$

The negative sign indicates a downward direction. We can assume that gravity is constant for the problems we will be examining, because we will be staying close to the surface of the earth. The acceleration of gravity decreases as an object moves very far from the earth. It is also different on other celestial bodies such as the Moon.

The equation that shows the height of an object in free fall is given by

$$y = \frac{1}{2}gt^2 + y_0$$

The term y_0 represents the initial height of the object t is time, and g is the force of gravity. There are two choices for the equation you can use.

$$\begin{aligned} y &= -4.9t^2 + y_0 && \text{If you wish to have the height in meters.} \\ y &= -16t^2 + y_0 && \text{If you wish to have the height in feet.} \end{aligned}$$

Example 8 Free fall

How long does it take a ball to fall from a roof to the ground 25 feet below?

Solution

$$\begin{aligned} \text{Since we are given the height in feet, use equation} \quad & y = -16t^2 + y_0 \\ \text{The initial height is } y_0 = 25\ feet, \text{ so} \quad & y = -16t^2 + 25 \\ \text{The height when the ball hits the ground is } y = 0, \text{ so} \quad & 0 = -16t^2 + 25 \\ \text{Solve for } t \quad & 16t^2 = 25 \\ & t^2 = \frac{25}{16} \\ & t = \frac{5}{4} \text{ or } t = -\frac{5}{4} \end{aligned}$$

We can discard the solution $t = -\frac{5}{4}$ since only positive values for time makes sense in this case,

Answer It takes the ball 1.25 seconds to fall to the ground.

Example 9 Free fall

A rock is dropped from the top of a cliff and strikes the ground 7.2 seconds later. How high is the cliff in meters?

Solution

$$\begin{aligned} \text{Since we want the height in meters, use equation} \quad & y = -4.9t^2 + y_0 \\ \text{The time of flight is } t = 7.2\ seconds \quad & y = -4.9(7.2)^2 + y_0 \\ \text{The height when the ball hits the ground is } y = 0, \text{ so} \quad & 0 = -4.9(7.2)^2 + y_0 \\ \text{Simplify} \quad & 0 = -254 + y_0 \text{ so } y_0 = 254 \end{aligned}$$

Answer The cliff is 254 meters high.

Example 10

Victor drops an apple out of a window on the 10^{th} *floor which is 120 feet above ground. One second later Juan drops an orange out of a* 6^{th} *floor window which is 72 feet above the ground. Which fruit reaches the ground first? What is the time difference between the fruits' arrival to the ground?*

Solution Let's find the time of flight for each piece of fruit.

For the Apple we have the following.

$$\begin{aligned} \text{Since we have the height in feet, use equation} \qquad & y = -16t^2 + y_0 \\ \text{The initial height } y_0 = 120\ feet. \qquad & y = -16t^2 + 120 \\ \text{The height when the ball hits the ground is } y = 0, \text{ so} \qquad & 0 = -16t^2 + 120 \\ \text{Solve for } t \qquad & 16t^2 = 120 \\ & t^2 = \frac{120}{16} = 7.5 \\ & t = 2.74 \text{ or } t = -2.74\ seconds. \end{aligned}$$

For the orange we have the following.

$$\begin{aligned} \text{The initial height } y_0 = 72\ feet. \qquad & 0 = -16t^2 + 72 \\ \text{Solve for } t. \qquad & 16t^2 = 72 \\ & t^2 = \frac{72}{16} = 4.5 \\ & t = 2.12 \text{ or } t = -2.12\ seconds \end{aligned}$$

But, don't forget that the orange was thrown out one second later, so add one second to the time of the orange. It hit the ground 3.12 seconds after Victor dropped the apple.

Answer The apple hits the ground first. It hits the ground 0.38 seconds before the orange. (Hopefully nobody was on the ground at the time of this experiment—don't try this one at home, kids!).

Review Questions

Solve the following quadratic equations.

1. $x^2 - 1 = 0$
2. $x^2 - 100 = 0$
3. $x^2 + 16 = 0$
4. $9x^2 - 1 = 0$
5. $4x^2 - 49 = 0$
6. $64x^2 - 9 = 0$
7. $x^2 - 81 = 0$
8. $25x^2 - 36 = 0$
9. $x^2 + 9 = 0$
10. $x^2 - 16 = 0$
11. $x^2 - 36 = 0$
12. $16x^2 - 49 = 0$
13. $(x-2)^2 = 1$
14. $(x+5)^2 = 16$

15. $(2x-1)^2-4=0$
16. $(3x+4)^2=9$
17. $(x-3)^2+25=0$
18. $x^2-6=0$
19. $x^2-20=0$
20. $3x^2+14=0$
21. $(x-6)^2=5$
22. $(4x+1)^2-8=0$
23. $x^2-10x+25=9$
24. $x^2+18x+81=1$
25. $4x^2-12x+9=16$
26. $(x+10)^2=2$
27. $x^2+14x+49=3$
28. $2(x+3)^2=8$
29. Susan drops her camera in the river from a bridge that is 400 feet high. How long is it before she hears the splash?
30. It takes a rock 5.3 seconds to splash in the water when it is dropped from the top of a cliff. How high is the cliff in meters?
31. Nisha drops a rock from the roof of a building 50 feet high. Ashaan drops a quarter from the top story window, 40 feet high, exactly half a second after Nisha drops the rock. Which hits the ground first?

Review Answers

1. $x=1, x=-1$
2. $x=10, x=-10$
3. No real solution.
4. $x=\frac{1}{3}, x=-\frac{1}{3}$
5. $x=\frac{7}{2}, x=-\frac{7}{2}$
6. $x=\frac{3}{8}, x=-\frac{3}{8}$
7. $x=9, x=-9$
8. $x=\frac{6}{5}, x=-\frac{6}{5}$
9. No real solution.
10. $x=4, x=-4$
11. $x=6, x=-6$
12. $x=\frac{7}{4}, x=-\frac{7}{4}$
13. $x=3, x=1$
14. $x=-1, x=-9$
15. $x=\frac{3}{2}, x=-\frac{1}{2}$
16. $x=-\frac{1}{3}, x=-\frac{7}{3}$
17. No real solution.
18. $x\approx 2.45, x\approx -2.45$
19. $x\approx 4.47, x\approx -4.47$
20. No real solution.
21. $x\approx 8.24, x\approx 3.76$
22. $x\approx 0.46, x\approx -0.96$
23. $x=8, x=2$
24. $x=-8, x=-10$
25. $x=\frac{7}{2}, x=-\frac{1}{2}$
26. $x\approx -8.59, x\approx -11.41$
27. $x\approx -5.27, x\approx -8.73$
28. $x=-1, x=-5$

29. $t = 5 \ seconds$
30. $y_0 = 137.6 \ meters$
31. .

3.4 Solving Quadratic Equations by Completing the Square

Learning objectives

- Complete the square of a quadratic expression.
- Solve quadratic equations by completing the square.
- Solve quadratic equations in standard form.
- Graph quadratic equations in vertex form.
- Solve real-world problems using functions by completing the square.

Introduction

You saw in the last section that if you have a quadratic equation of the form

$(x-2)^2 = 5$

We can easily solve it by taking the square root of each side.

$x-2 = \sqrt{5}$ and $x-2 = -\sqrt{5}$

Then simplify and solve.

$x = 2+\sqrt{5} \approx 4.24$ and $x = 2-\sqrt{5} \approx -0.24$

Unfortunately, quadratic equations are not usually written in this nice form. In this section, you will learn the method of **completing the squares** in which you take any quadratic equation and rewrite it in a form so that you can take the square root of both sides.

Complete the Square of a Quadratic Expression

The purpose of the method of completing the squares is to rewrite a quadratic expression so that it contains a perfect square trinomial that can be factored as the square of a binomial. Remember that the square of a binomial expands. Here is an example of this.

$$(x+a)^2 = x^2+2ax+a^2$$
$$(x-a)^2 = x^2-2ax+a^2$$

In order to have a perfect square trinomial, we need two terms that are perfect squares and one term that is twice the product of the square roots of the other terms.

Example 1

Complete the square for the quadratic expression x^2+4x.

Solution To complete the square, we need a constant term that turns the expression into a perfect square trinomial. Since the middle term in a perfect square trinomial is always two times the product of the square roots of the other two terms, we rewrite our expression as

$$x^2+2(2)(x)$$

We see that the constant we are seeking must be 2^2.

$$x^2+2(2)(x)+2^2$$

Answer By adding 4, this can be factored as: $(x+2)^2$

BUT, we just changed the value of this expression $x^2+4x \neq (x+2)^2$. Later we will show how to account for this problem. You need to add and subtract the constant term.

Also, this was a relatively easy example because a, the coefficient of the x^2 term was 1. If $a \neq 1$, we must factor a from the whole expression before completing the square.

Example 2

Complete the square for the quadratic expression $4x^2+32x$

Solution

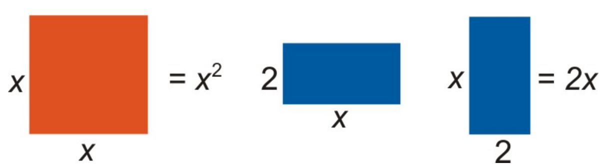
Factor the coefficient of the x^2 term.	$4(x^2+8x)$
Now complete the square of the expression in parentheses then rewrite the expression.	$4(x^2+2(4)(x))$
We complete the square by adding the constant 4^2.	$4(x^2+2(4)(x)+4^2)$
Factor the perfect square trinomial inside the parenthesis.	$4(x+4)^2$

Our answer is $4(x+4)^2$.

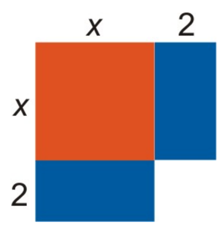
The expression **"completing the square"** comes from a geometric interpretation of this situation. Let's revisit the quadratic expression in Example 1.

$$x^2+4x$$

We can think of this expression as the sum of three areas. The first term represents the area of a square of side x. The second expression represents the areas of two rectangles with a length of 2 and a width of x:

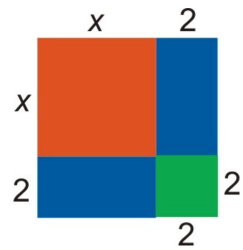


We can combine these shapes as follows



We obtain a square that is not quite complete.

In order to complete the square, we need a square of side 2.



We obtain a square of side $x+2$.

The area of this square is: $(x+2)^2$.

You can see that completing the square has a geometric interpretation.

Finally, here is the algebraic procedure for completing the square.

$$\begin{aligned}
x^2+bx+c&=0\\
x^2+bx&=-c\\
x^2+bx+\left(\frac{b}{2}\right)^2&=-c+\left(\frac{b}{2}\right)^2\\
\left(x+\frac{b}{2}\right)^2&=-c+\left(\frac{b}{2}\right)^2
\end{aligned}$$

Solve Quadratic Equations by Completing the Square

Let's demonstrate the method of **completing the square** with an example.

Example 3

Solve the following quadratic equation $x^2+12x=3$.

Solution

The method of completing the square is as follows.

1. Rewrite as $x^2+2(6)x=3$

2. In order to have a perfect square trinomial on the right-hand-side we need to add the constant 6^2. Add this constant to both sides of the equation.

$$x^2+2(6)(x)+6^2=3+6^2$$

3. Factor the perfect square trinomial and simplify the right hand side of the equation.

$$(x+6)^2=39$$

4. Take the square root of both sides.

$$\begin{aligned}
x+6&=\sqrt{39} && \text{and} && x+6=-\sqrt{39}\\
x&=-6+\sqrt{39}\approx 0.24 && \text{and} && x=-6-\sqrt{39}\approx -12.24
\end{aligned}$$

Answer $x = 0.24$ and $x = -12.24$

If the coefficient of the x^2 term is not one, we must divide that number from the whole expression before completing the square.

Example 4

Solve the following quadratic equation $3x^2 - 10x = -1$.

Solution:

1. Divide all terms by the coefficient of the x^2 term.

$$x^2 - \frac{10}{3}x = -\frac{1}{3}$$

2. Rewrite as

$$x^2 - 2\left(\frac{5}{3}\right)(x) = -\frac{1}{3}$$

3. In order to have a perfect square trinomial on the right hand side we need to add the constant $\left(\frac{5}{3}\right)^2$. Add this constant to both sides of the equation.

$$x^2 - 2\left(\frac{5}{3}\right)(x) + \left(\frac{5}{3}\right)^2 = -\frac{1}{3} + \left(\frac{5}{3}\right)^2$$

4. Factor the perfect square trinomial and simplify.

$$\left(x - \frac{5}{3}\right)^2 = \frac{1}{3} + \frac{25}{9}$$
$$\left(x - \frac{5}{3}\right)^2 = \frac{22}{9}$$

5. Take the square root of both sides.

$$x - \frac{5}{3} = \sqrt{\frac{22}{9}} \quad \text{and} \quad x - \frac{5}{3} = -\sqrt{\frac{22}{9}}$$
$$x = \frac{5}{3} + \sqrt{\frac{22}{9}} \approx 3.23 \quad \text{and} \quad x = \frac{5}{3} - \sqrt{\frac{22}{9}} \approx 0.1$$

Answer $x = 3.23$ and $x = 0.1$

Solve Quadratic Equations in Standard Form

An equation in standard form is written as $ax^2 + bx + c = 0$. We solve an equation in this form by the method of completing the square. First we move the constant term to the right hand side of the equation.

Example 5

Solve the following quadratic equation $x^2+15x+12=0$.

Solution

The method of completing the square is as follows:

1. Move the constant to the other side of the equation.

$$x^2+15x=-12$$

2. Rewrite as

$$x^2+2\left(\frac{15}{2}\right)(x)=-12$$

3. Add the constant $\left(\frac{15}{2}\right)^2$ to both sides of the equation

$$x^2+2\left(\frac{15}{2}\right)(x)+\left(\frac{15}{2}\right)^2=-12+\left(\frac{15}{2}\right)^2$$

4. Factor the perfect square trinomial and simplify.

$$\left(x+\frac{15}{2}\right)^2=-12+\frac{225}{4}$$
$$\left(x+\frac{15}{2}\right)^2=\frac{177}{4}$$

5. Take the square root of both sides.

$$x+\frac{15}{2}=\sqrt{\frac{177}{4}} \quad \text{and} \quad x+\frac{15}{2}=-\sqrt{\frac{177}{4}}$$
$$x+-\frac{15}{2}+\sqrt{\frac{177}{4}}\approx-0.85 \quad \text{and} \quad x+-\frac{15}{2}+\sqrt{\frac{177}{4}}\approx-14.15$$

Answer $x=-0.85$ and $x=-14.15$

Graph Quadratic Functions in Vertex Form

Probably one of the best applications of the method of completing the square is using it to rewrite a quadratic function in vertex form.

The vertex form of a quadratic function is $y-k=a(x-h)^2$.

This form is very useful for graphing because it gives the vertex of the parabola explicitly. The vertex is at point (h,k).

It is also simple to find the $x-$intercepts from the vertex from by setting $y=0$ and taking the square root of both sides of the resulting equation.

The $y-$intercept can be found by setting $x = 0$ and simplifying.

Example 6

Find the vertex, the $x-$intercepts and the $y-$intercept of the following parabolas.

(a) $y-2 = (x-1)^2$

(b) $y+8 = 2(x-3)^2$

Solution

a) $y-2 = (x-1)^2$

Vertex is (1, 2)

To find $x-$intercepts,

$$\begin{array}{rlcl} \text{Set } y = 0 & -2 = (x-1)^2 & & \\ \text{Take the square root of both sides} & \sqrt{-2} = x-1 & \text{and} & -\sqrt{-2} = x-1 \end{array}$$

The solutions are not real (because you cannot take the square root of a negative number), so there are **no** $x-$intercepts.

To find $y-$intercept,

$$\begin{array}{ll} \text{Set } x = 0 & y-2 = (-1)^2 \\ \text{Simplify} & y-2 = 1 \Rightarrow y = 3 \end{array}$$

b) $y+8 = 2(x-3)^2$

$$\begin{array}{ll} \text{Rewrite} & y-(-8) = 2(x-3)^2 \\ \text{Vertex is} & (3,-8) \end{array}$$

To find $x-$intercepts,

$$\begin{array}{rlcl} \text{Set } y = 0: & 8 = 2(x-3)^2 & & \\ \text{Divide both sides by 2.} & 4 = (x-3)^2 & & \\ \text{Take the square root of both sides}: & 2 = x-3 & \text{and} & -2 = x-3 \\ \text{Simplify}: & x = 5 & \text{and} & x = 1 \end{array}$$

To find the $y-$intercept,

$$\begin{array}{ll} \text{Set } x = 0. & y+8 = 2(-3)^2 \\ \text{Simplify}: & y+8 = 18 \Rightarrow y = 10 \end{array}$$

To graph a parabola, we only need to know the following information.

- The coordinates of the vertex.

- The $x-$intercepts.
- The $y-$intercept.
- Whether the parabola turns up or down. Remember that if $a > 0$, the parabola turns up and if $a < 0$ then the parabola turns down.

Example 7

Graph the parabola given by the function $y + 1 = (x+3)^2$.

Solution

Rewrite.	$y-(-1) = (x-(-3))^2$
Vertex is	$(-3,-1)$

To find the $x-$intercepts

Set $y = 0$	$1 = (x+3)^2$		
Take the square root of both sides	$1 = x+3$	and	$-1 = x+3$
Simplify	$x = -2$	and	$x = -4$

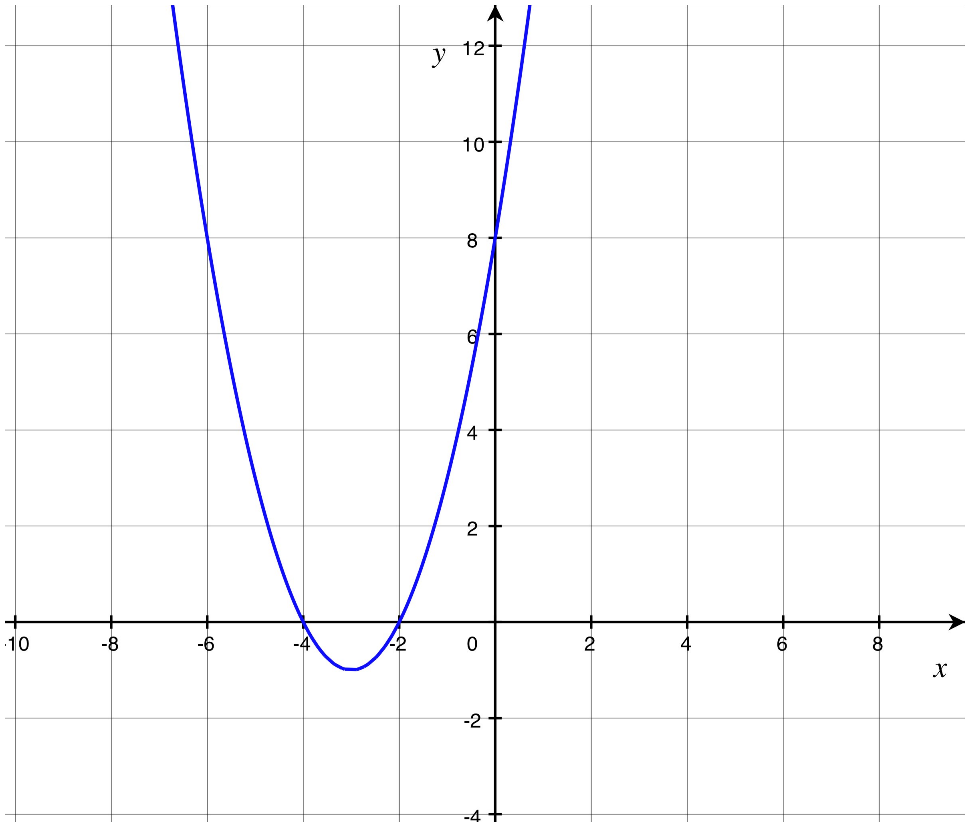
$x-$**intercepts:** (-2, 0) and (-4, 0)

To find the $y-$intercept

Set $x = 0$	$y+1(3)^2$	
Simplify	$y = 8$	$y-\text{intercept}: (0,8)$

Since $a > 0$, the parabola **turns up**.

Graph all the points and connect them with a smooth curve.



Example 8

Graph the parabola given by the function $y = -\frac{1}{2}(x-2)^2$

Solution:

Re-write	$y-(0) = -\frac{1}{2}(x-2)^2$
Vertex is	$(2,0)$

To find the $x-$intercepts,

Set $y = 0$.	$0 = -\frac{1}{2}(x-2)^2$
Multiply both sides by -2.	$0 = (x-2)^2$
Take the square root of both sides.	$0 = x-2$
Simplify.	$x = 2$

$x-$**intercept** (2, 0)

Note: there is only one $x-$intercept, indicating that the vertex is located at this point (2, 0).

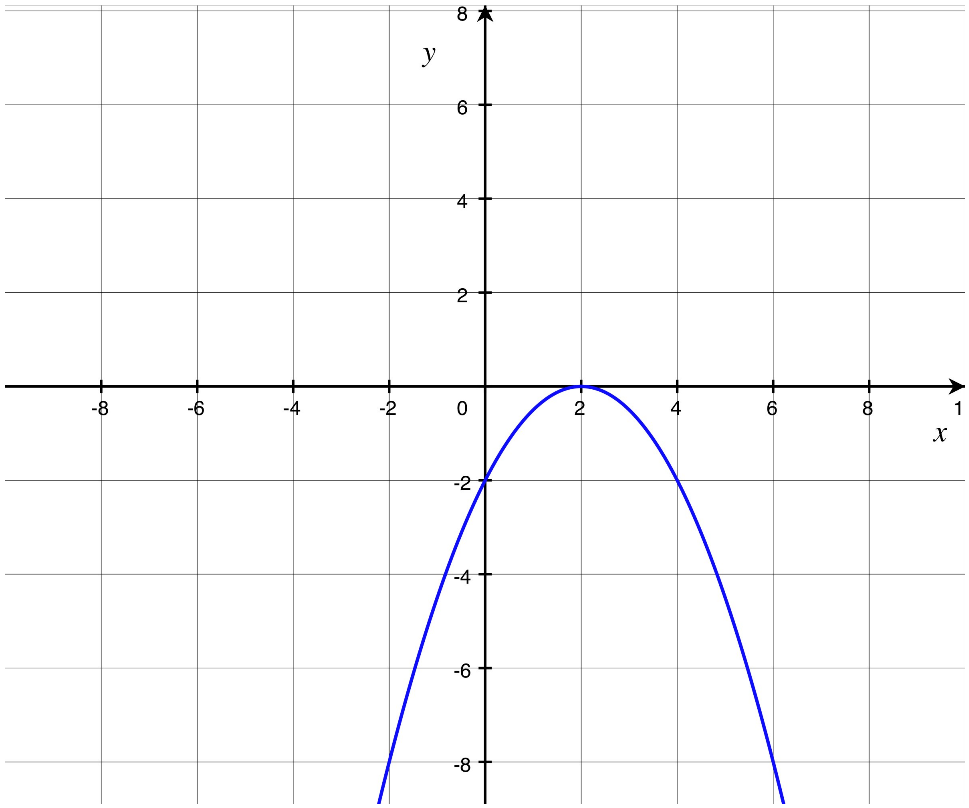
To find the $y-$intercept

Set $x = 0$	$y = -\frac{1}{2}(-2)^2$
Simplify	$y = -\frac{1}{2}(4) \Rightarrow y = -2$

$y-$**intercept** (0, -2)

Since $a < 0$, the parabola **turns down.**

Graph all the points and connect them with a smooth curve.



Solve Real-World Problems Using Quadratic Functions by Completing the Square

Projectile motion with vertical velocity

In the last section you learned that an object that is dropped falls under the influence of gravity. The equation for its height with respect to time is given by

$$y = \frac{1}{2}gt^2 + y_0$$

The term y_0 represents the initial height of the object and the coefficient of gravity on earth is given by

$g = -9.8\ m/s^2$ or $g = -32\ ft/s^2$.

On the other hand, if an object is thrown straight up or straight down in the air, it has an initial vertical velocity. This term is usually represented by the notation v_{0y}. Its value is positive if the object is thrown up in the air, and, it is negative if the object is thrown down. The equation for the height of the object in this case is given by the equation

$$y = \frac{1}{2}gt^2 + v_{0y}t + y_0$$

There are two choices for the equation to use in these problems.

$y = -4.9t^2 + v_{0y}t + y_0$	If you wish to have the height in meters.
$y = -16t^2 + v_{0y}t + y_0$	If you wish to have the height in feet.

Example 9

An arrow is shot straight up from a height of 2 meters with a velocity of 50 m/s.

a) How high will an arrow be four seconds after being shot? After eight seconds?

b) At what time will the arrow hit the ground again?

c) What is the maximum height that the arrow will reach and at what time will that happen?

Solution

Since we are given the velocity in meters per second, use the equation $y = -4.9t^2 + v_{0y}t + y_0$

We know $v_{0y} = 50\ m/s$ and $y_0 = 2\ meters$ so, $y = -4.9t^2 + 50t + 2$

a) To find how high the arrow will be 4 seconds after being shot we substitute 4 for t

$$\begin{aligned} y &= -4.9(4)^2 + 50(4) + 2 \\ &= -4.9(16) + 200 + 2 = 123.6\ meters \end{aligned}$$

we substitute $t = 8$

$$\begin{aligned} y &= -4.9(8)^2 + 50(8) + 2 \\ &= -4.9(64) + 400 + 5 = 88.4\ meters \end{aligned}$$

b) The height of the arrow on the ground is $y = 0$, so $0 = -4.9t^2 + 50t + 2$

Solve for t by completing the square	$-4.9t^2 + 50t = -2$
Factor the	$-4.9 - 4.9(t^2 - 10.2t) = -2$
Divide both sides by	$-4.9t^2 - 10.2t = 0.41$
Add 5.1^2 to both sides	$t^2 - 2(5.1)t + (5.1)^2 = 0.41 + (5.1)^2$
Factor	$(t - 5.1)^2 = 26.43$
Solve	$t - 5.1 \approx 5.14$ and $t - 5.1 \approx -5.14$
	$t \approx 10.2\ sec$ and $t \approx -0.04\ sec$

c) If we graph the height of the arrow with respect to time, we would get an upside down parabola $(a < 0)$. The maximum height and the time when this occurs is really the vertex of this parabola (t, h).

We rewrite the equation in vertex form.	$y = -4.9t^2 + 50t + 2$
	$y - 2 = -4.9t^2 + 50t$
	$y - 2 = -4.9(t^2 - 10.2t)$
Complete the square inside the parenthesis.	$y - 2 - 4.9(5.1)^2 = -4.9(t^2 - 10.2t + (5.1)^2)$
	$y - 129.45 = -4.9(t - 5.1)^2$

The vertex is at (5.1, 129.45). In other words, **when** $t = 5.1$ *seconds*, **the height is** $y = 129$ *meters*.

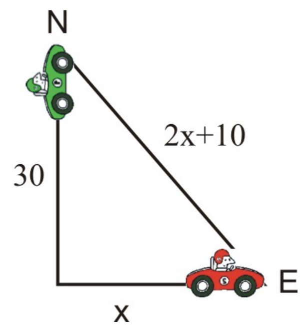
Another type of application problem that can be solved using quadratic equations is one where two objects are moving away in directions perpendicular from each other. Here is an example of this type of problem.

Example 10

Two cars leave an intersection. One car travels north; the other travels east. When the car traveling north had gone 30 miles, the distance between the cars was 10 miles more than twice the distance traveled by the car heading east. Find the distance between the cars at that time.

Solution

Let x = the distance traveled by the car heading east.



$2x+10$ = the distance between the two cars

Let's make a sketch

We can use the Pythagorean Theorem $(a^2+b^2=c^2)$ to find an equation for x:

$$x^2+30^2=(2x+10)^2$$

Expand parentheses and simplify.

$$\begin{aligned} x^2+900&=4x^2+40x+100\\ 800&=3x^2+40x \end{aligned}$$

Solve by completing the square.

$$\begin{aligned} \frac{800}{3}&=x^2+\frac{40}{3}x\\ \frac{800}{3}+\left(\frac{20}{3}\right)^4&=x^2+2\left(\frac{20}{3}\right)x+\left(\frac{20}{3}\right)^3\\ \frac{2800}{9}&=\left(x+\frac{20}{3}\right)^2\\ x+\frac{20}{3}\approx 17.6 &\text{ and } x+\frac{20}{3}\approx -17.6\\ x\approx 11 &\text{ and } x\approx -24.3 \end{aligned}$$

Since only positive distances make sense here, the distance between the two cars is $2(11)+10=32$ *miles*.

Answer The distance between the two cars is 32 miles.

Review Questions

Complete the square for each expression.

1. $x^2 + 5x$
2. $x^2 - 2x$
3. $x^2 + 3x$
4. $x^2 - 4x$
5. $3x^2 + 18x$
6. $2x^2 - 22x$
7. $8x^2 - 10x$
8. $5x^2 + 12x$

Solve each quadratic equation by completing the square.

9. $x^2 - 4x = 5$

10. $x^2 - 5x = 10$

11. $x^2 + 10x + 15 = 0$

12. $x^2 + 4x + 16 = 0$

13. $2x^2 - 18x = 0$

14. $4x^2 + 5x = -1$

15. $10x^2 - 30x - 8 = 0$

16. $5x^2 + 15x - 40 = 0$

Rewrite each quadratic function in vertex form.

17. $y = x^2 - 6x$

18. $y + 1 = -2x^2 - x$

19. $y = 9x^2 + 3x - 10$

20. $y = 32x^2 + 60x + 10$ For each parabola, find

    a. The vertex

    b. $x-$intercepts

    c. $y-$intercept

    d. If it turns up or down.

    e. The graph the parabola.

21. $y - 4 = x^2 + 8x$

22. $y = -4x^2 + 20x - 24$

23. $y = 3x^2 + 15x$

24. $y + 6 = -x^2 + x$

25. Sam throws an egg straight down from a height of 25 feet. The initial velocity of the egg is 16 ft/sec. How long does it take the egg to reach the ground?

26. Amanda and Dolvin leave their house at the same time. Amanda walks south and Dolvin bikes east. Half an hour later they are 5.5 miles away from each other and Dolvin has covered three miles more than the distance that Amanda covered. How far did Amanda walk and how far did Dolvin bike?

Review Answers

1. $x^2+5x+\frac{25}{4}=\left(x+\frac{5}{2}\right)^2$
2. $x^2-2x+1=(x-1)^2$
3. $x^2+3x+\frac{9}{4}=\left(x+\frac{3}{2}\right)^2$
4. $x^2-4x+4=(x-2)^2$
5. $3(x^2+6x+9)=3(x+3)^2$
6. $2\left(x^2-11x+\frac{121}{4}\right)=2\left(x-\frac{11}{2}\right)^2$
7. $8\left(x^2-\frac{5}{4}x+\frac{25}{64}\right)=8\left(x-\frac{5}{8}\right)^2$
8. $5\left(x^2+\frac{12}{5}x+\frac{36}{25}\right)=5\left(x+\frac{6}{5}\right)^2$
9. $5,-1$
10. $\frac{5}{2}\pm\frac{\sqrt{65}}{2}$
11. $-5\pm\sqrt{10}$
12. $-2\pm i\sqrt{3}$
13. $0,9$
14. $-1,-.25$
15. $-3.25,-.25$
16. $-\frac{3}{2}\pm\sqrt{41}$
17. $y+9=(x-3)^2$
18. $y+\frac{7}{8}=-2\left(x+\frac{1}{4}\right)^2$
19. $y+10.25=9\left(x+\frac{1}{6}\right)^2$
20. $y-\frac{305}{8}=-32\left(x-\frac{15}{16}\right)^2$
21. $y+12=(x+4)^2$; vertex (-4, -12); $x-$intercepts (-7.46, 0), (-.54, 0); $y-$intercept (0, 4); turns up.
22. $y-1=-4\left(x-\frac{5}{2}\right)^2$; vertex (2.5, 1); $x-$intercepts (2, 0), (3, 0) $y-$intercept (0, -24); turns down.
23. $y+18.75=3(x+2.5)^2$; vertex (-2.5, -18.75); $x-$intercepts (0, 0), (-5, 0); $y-$intercept (0, 0); turns up.
24. $y+\frac{23}{4}=-\left(x-\frac{1}{2}\right)^2$; vertex (0.5, -5.75); $x-$intercepts none; $y-$intercept (0, -6); turns down.
25. 0.85 *seconds*
26. Amanda 2.1 *miles*, Dolvin 5.1 *miles*

3.5 Solving Quadratic Equations by the Quadratic Formula

Learning objectives

- Solve quadratic equations using the quadratic formula.
- Identify and choose methods for solving quadratic equations.
- Solve real-world problems using functions by completing the square.

Introduction

In this section, you will solve quadratic equations using the **Quadratic Formula.** Most of you are already familiar with this formula from previous mathematics courses. It is probably the most used method for solving quadratic equations. For a quadratic equation in standard form

$$ax^2+bx+c=0$$

The solutions are found using the following formula.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

We will start by explaining where this formula comes from and then show how it is applied. This formula is derived by solving a general quadratic equation using the method of completing the square that you learned in the previous section.

We start with a general quadratic equation.	$ax^2+bx+c=0$
Subtract the constant term from both sides.	$ax^2+bx=-c$
Divide by the coefficient of the x^2 term.	$x^2+\frac{b}{a}x=-\frac{c}{a}$
Rewrite.	$x^2+2\left(\frac{b}{2a}\right)x=-\frac{c}{a}$
Add the constant $\left(\frac{b}{2a}\right)^2$ to both sides.	$x^2+2\left(\frac{b}{2a}\right)x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$
Factor the perfect square trinomial.	$\left(x+\frac{b}{2a}\right)^2=-\frac{4ac}{4a^2}+\frac{b^2}{4a^2}$
Simplify.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$
Take the square root of both sides.	$x+\frac{b}{2a}=\sqrt{\frac{b^2-4ac}{4a^2}}$ and $x+\frac{b}{2a}=-\sqrt{\frac{b^2-4ac}{4a^2}}$
Simplify.	$x+\frac{b}{2a}=\sqrt{\frac{b^2-4ac}{2a}}$ and $x+\frac{b}{2a}=-\sqrt{\frac{b^2-4ac}{2a}}$
	$x=-\frac{b}{2a}+\sqrt{\frac{b^2-4ac}{2a}}$
	$x=-\frac{b}{2a}-\sqrt{\frac{b^2-4ac}{2a}}$
	$x=\frac{-b+\sqrt{b^2-4ac}}{2a}$
	$x=\frac{-b-\sqrt{b^2-4ac}}{2a}$

This can be written more compactly as $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

You can see that the familiar formula comes directly from applying the method of completing the square. Applying the method of completing the square to solve quadratic equations can be tedious. The quadratic formula is a more straightforward way of finding the solutions.

Solve Quadratic Equations Using the Quadratic Formula

Applying the quadratic formula basically amounts to plugging the values of a, b and c into the quadratic formula.

Example 1

Solve the following quadratic equation using the quadratic formula.

a) $2x^2+3x+1=0$

b) $x^2-6x+5=0$

c) $-4x^2+x+1=0$

Solution

Start with the quadratic formula and plug in the values of a, b and c.

a)

$$\begin{aligned} &\text{Quadratic formula} && x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &\text{Plug in the values } a=2, b=3, c=1. && x=\frac{-3\pm\sqrt{(3)^2-4(2)(1)}}{2(2)} \\ &\text{Simplify.} && x=\frac{-3\pm\sqrt{9-8}}{4}=\frac{-3\pm\sqrt{1}}{4} \\ &\text{Separate the two options.} && x=\frac{-3+1}{4} \text{ and } x=\frac{-3-1}{4} \\ &\text{Solve.} && x=\frac{-2}{4}=-\frac{1}{2} \text{ and } x=\frac{-4}{4}=-1 \end{aligned}$$

Answer $x=-\frac{1}{2}$ and $x=-1$

b)

$$\begin{aligned} &\text{Quadratic formula.} && x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &\text{Plug in the values } a=1, b=-6, c=5. && x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(5)}}{2(1)} \\ &\text{Simplify.} && x=\frac{6\pm\sqrt{36-20}}{2}=\frac{6\pm\sqrt{16}}{2} \\ &\text{Separate the two options.} && x=\frac{6+4}{2} \text{ and } x=\frac{6-4}{2} \\ &\text{Solve} && x=\frac{10}{2}=5 \text{ and } x=\frac{2}{2}=1 \end{aligned}$$

Answer $x=5$ and $x=1$

c)

$$\begin{aligned} &\text{Quadratic formula.} && x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &\text{Plug in the values } a=-4, b=1, c=1. && x=\frac{-1\pm\sqrt{(1)^2-4(-4)(1)}}{2(-4)} \\ &\text{Simplify.} && x=\frac{-1\pm\sqrt{1+16}}{-8}=\frac{-1\pm\sqrt{17}}{-8} \\ &\text{Separate the two options.} && x=\frac{-1+\sqrt{17}}{-8} \text{ and } x=\frac{-1-\sqrt{17}}{-8} \\ &\text{Solve.} && x\approx-.39 \text{ and } x\approx .64 \end{aligned}$$

Answer $x\approx-.39$ and $x\approx .64$

Often when we plug the values of the coefficients into the quadratic formula, we obtain a negative number inside the square root. Since the square root of a negative number does not give real answers, we say that the equation has no real solutions. In more advanced mathematics classes, you will learn how to work with "complex" (or "imaginary") solutions to quadratic equations.

Example 2

Solve the following quadratic equation using the quadratic formula $x^2+2x+7=0$

Solution:

a)

$$\begin{aligned} &\text{Quadratic formula.} && x=\frac{b\pm\sqrt{b^2-4ac}}{2a} \\ &\text{Plug in the values } a=1,b=2,c=7. && x=\frac{-2\pm\sqrt{(2)^2-4(1)(7)}}{2(1)} \\ &\text{Simplify.} && x=\frac{-2\pm\sqrt{4-28}}{2}=\frac{-2\pm\sqrt{-24}}{2} \end{aligned}$$

Answer There are no real solutions.

To apply the quadratic formula, we must make sure that the equation is written in standard form. For some problems, we must rewrite the equation before we apply the quadratic formula.

Example 3

Solve the following quadratic equation using the quadratic formula.

a) $x^2-6x=10$

b) $8x^2=5x+6$

Solution:

a)

$$\begin{aligned} &\text{Rewrite the equation in standard form.} && x^2-6x-10=0 \\ &\text{Quadratic formula} && x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &\text{Plug in the values } a=1,b=-6,c=-10. && x=\frac{-(-6)\pm\sqrt{(-6)^2-4(1)(-10)}}{2(1)} \\ &\text{Simplify.} && x=\frac{6\pm\sqrt{36+40}}{2}=\frac{6\pm\sqrt{76}}{2} \\ &\text{Separate the two options.} && x=\frac{6+\sqrt{76}}{2} \text{ and } x=\frac{6-\sqrt{76}}{2} \\ &\text{Solve.} && x\approx 7.36 \text{ and } x\approx -1.36 \end{aligned}$$

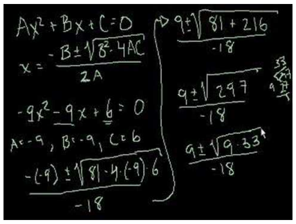
Answer $x\approx 7.36$ and $x\approx -1.36$

b)

$$\begin{aligned} &\text{Rewrite the equation in standard form.} && 8x^2+5x+6=0 \\ &\text{Quadratic formula} && x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \\ &\text{Plug in the values } a=8,b=5,c=6. && x=\frac{-5\pm\sqrt{(5)^2-4(8)(6)}}{2(8)} \\ &\text{Simplify.} && x=\frac{-5\pm\sqrt{25-192}}{16}=\frac{-5\pm\sqrt{-167}}{16} \end{aligned}$$

Answer no real solutions

Multimedia Link For more examples of solving quadratic equations using the quadratic formula, see KhanAcademyEquation Part 2 (9:14)



MEDIA

Click image to the left for more content.

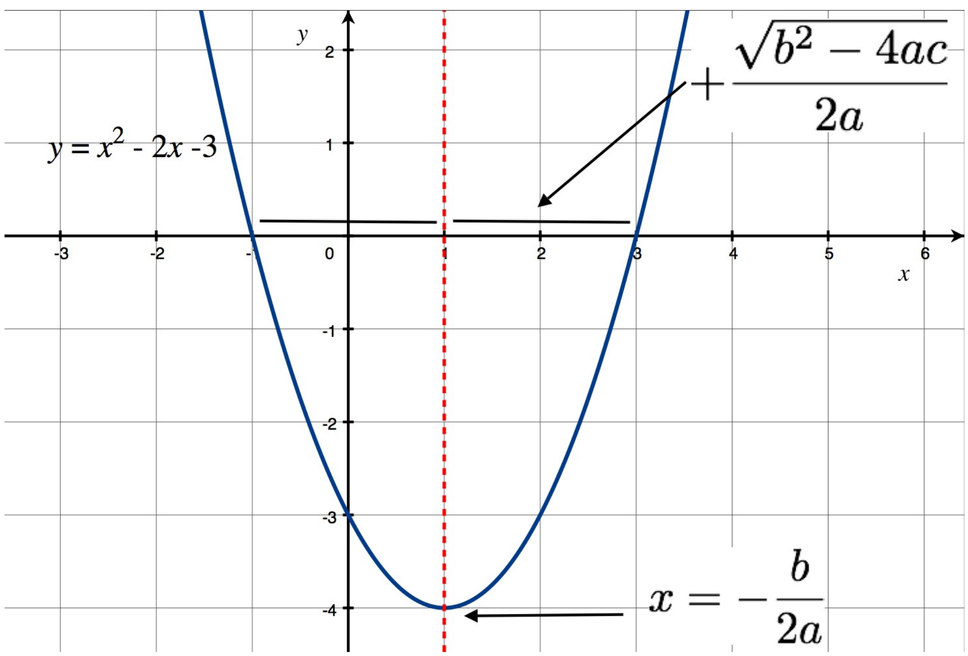
. This video is not necessarily different from the examples above, but it does help reinforce the procedure of using the quadratic formula to solve equations.

Finding the Vertex of a Parabola with the Quadratic Formula

Sometimes you get more information from a formula beyond what you were originally seeking. In this case, the quadratic formula also gives us an easy way to locate the vertex of a parabola.

First, recall that the quadratic formula tells us the **roots** or **solutions** of the equation $ax^2+bx+c=0$. Those roots are

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$



We can rewrite the fraction in the quadratic formula as

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}.$$

Recall that the roots are **symmetric** about the vertex. In the form above, we can see that the roots of a quadratic equation are symmetric around the $x-$coordinate $-\frac{b}{2a}$ because they move $\frac{\sqrt{b^2-4ac}}{2a}$ units to the left and right (recall the $\pm$ sign) from the vertical line $x=-\frac{b}{2a}$. The image to the right illustrates this for the equation $x^2-2x-3=0$. The roots, -1 and 3 are both 2 units from the vertical line $x=1$.

Identify and Choose Methods for Solving Quadratic Equations.

In mathematics, you will need to solve quadratic equations that describe application problems or that are part of more complicated problems. You learned four ways of solving a quadratic equation.

- Factoring.
- Taking the square root.
- Completing the square.
- Quadratic formula.

Usually you will not be told which method to use. You will have to make that decision yourself. However, here are some guidelines to which methods are better in different situations.

Factoring is always best if the quadratic expression is easily factorable. It is always worthwhile to check if you can factor because this is the fastest method. Many expressions are not factorable so this method is not used very often in practice.

Taking the square root is best used when there is no x term in the equation.

Completing the square can be used to solve any quadratic equation. This is usually not any better than using the quadratic formula (in terms of difficult computations), however it is a very important method for re-writing a quadratic function in vertex form. It is also be used to re-write the equations of circles, ellipses and hyperbolas in standard form (something you will do in algebra II, trigonometry, physics, calculus, and beyond…).

Quadratic formula is the method that is used most often for solving a quadratic equation. When solving directly by taking square root and factoring does not work, this is the method that most people prefer to use.

If you are using factoring or the quadratic formula make sure that the equation is in standard form.

Example 4

Solve each quadratic equation

a) $x^2-4x-5=0$

b) $x^2=8$

c) $-4x^2+x=2$

d) $25x^2-9=0$

e) $3x^2=8x$

Solution

a) This expression if easily factorable so we can factor and apply the zero-product property:

$$\begin{aligned} \text{Factor.} \quad & (x-5)(x+1)=0 \\ \text{Apply zero-product property.} \quad & x-5=0 \text{ and } x+1=0 \\ \text{Solve.} \quad & x=5 \text{ and } x=-1 \end{aligned}$$

Answer $x=5$ and $x=-1$

b) Since the expression is missing the x term we can take the square root:

$$\text{Take the square root of both sides.} \quad x=\sqrt{8} \text{ and } x=-\sqrt{8}$$

Answer $x=2.83$ and $x=-2.83$

c) Rewrite the equation in standard form.

It is not apparent right away if the expression is factorable, so we will use the quadratic formula.

Quadratic formula $\quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Plug in the values $a = -4, b = 1, c = -2$. $\quad x = \frac{-1 \pm \sqrt{1^2 - 4(-4)(-2)}}{2(-4)}$

Simplify. $\quad x = \frac{-1 \pm \sqrt{1-32}}{-8} = \frac{-1 \pm \sqrt{-31}}{-8}$

Answer no real solution

d) This problem can be solved easily either with factoring or taking the square root. Let's take the square root in this case.

Add 9 to both sides of the equation. $\quad 25x^2 = 9$

Divide both sides by 25. $\quad x^2 = \frac{9}{25}$

Take the square root of both sides. $\quad x = \sqrt{\frac{9}{25}}$ and $x = -\sqrt{\frac{9}{25}}$

Simplify. $\quad x = \frac{3}{5}$ and $x = -\frac{3}{5}$

Answer $x = \frac{3}{5}$ and $x = -\frac{3}{5}$

e)

Rewrite the equation in standard form $\quad 3x^2 - 8x = 0$

Factor out common x term. $\quad x(3x - 8) = 0$

Set both terms to zero. $\quad x = 0$ and $3x = 8$

Solve. $\quad x = 0$ and $x = \frac{8}{3} = 2.67$

Answer $x = 0$ and $x = 2.67$

Solve Real-World Problems Using Quadratic Functions by any Method

Here are some application problems that arise from number relationships and geometry applications.

Example 5

The product of two positive consecutive integers is 156. Find the integers.

Solution

For two consecutive integers, one integer is one more than the other one.

Define

Let x = the smaller integer

$x + 1$ = the next integer

Translate

The product of the two numbers is 156. We can write the equation:

$$x(x+1) = 156$$

Solve

$$\begin{aligned} x^2 + x &= 156 \\ x^2 + x - 156 &= 0 \end{aligned}$$

Apply the quadratic formula with $a = 1, b = 1, c = -156$

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1^2 - 4(1)(-156)}}{2(1)} \\ x &= \frac{-1 \pm \sqrt{625}}{2} = \frac{-1 \pm 25}{2} \\ x &= \frac{-1+25}{2} \text{ and } x = \frac{-1-25}{2} \\ x &= \frac{24}{2} = 12 \text{ and } x = \frac{-26}{2} = -13 \end{aligned}$$

Since we are looking for positive integers take, $x = 12$

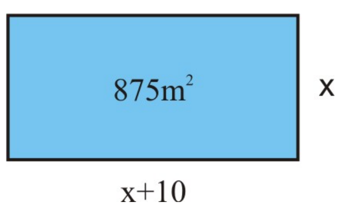
Answer 12 and 13

Check $12 \times 13 = 156$. The answer checks out.

Example 6

The length of a rectangular pool is 10 meters more than its width. The area of the pool is 875 square/meters. Find the dimensions of the pool.

Solution:



Draw a sketch

Define

Let x = the width of the pool

$x + 10$ = the length of the pool

Translate

The area of a rectangle is A = length $\times$ width, so

$$x(x+10) = 875$$

Solve

$$x^2 + 10x = 875$$
$$x^2 + 10x - 875 = 0$$

Apply the quadratic formula with $a = 1, b = 10$ and $c = -875$

$$x = \frac{-10 \pm \sqrt{(10)^2 - 4(1)(-875)}}{2(1)}$$
$$x = \frac{-10 \pm \sqrt{100 + 3500}}{2}$$
$$x = \frac{-10 \pm \sqrt{3600}}{2} = \frac{-10 \pm 60}{2}$$
$$x = \frac{-10 + 60}{2} \text{ and } x = \frac{-10 - 60}{2}$$
$$x = \frac{50}{2} = 25 \text{ and } x = \frac{-70}{2} = -35$$

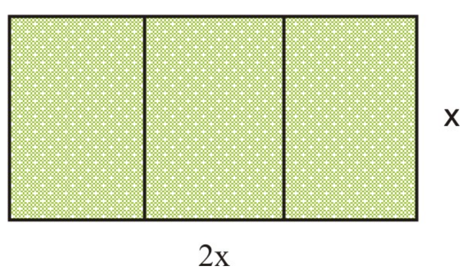
Since the dimensions of the pools should be positive, then $x = 25$ *meters*.

Answer The pool is 25 *meters* $\times$ 35 *meters*.

Check $25 \times 35 = 875\ m^2$. **The answer checks out.**

Example 7

Suzie wants to build a garden that has three separate rectangular sections. She wants to fence around the whole garden and between each section as shown. The plot is twice as long as it is wide and the total area is 200 ft^2. *How much fencing does Suzie need?*



Solution

Draw a Sketch

Define

Let x = the width of the plot

$2x$ = the length of the plot

Translate

Area of a rectangle is A = length $\times$ width, so

$$x(2x) = 200$$

Solve

$$2x^2 = 200$$

Solve by taking the square root.

$$\begin{aligned} x^2 &= 100 \\ x &= \sqrt{100} \text{ and } x = -\sqrt{100} \\ x &= 10 \text{ and } x = -10 \end{aligned}$$

We take $x = 10$ since only positive dimensions make sense.

The plot of land is $10\ feet \times 20\ feet$.

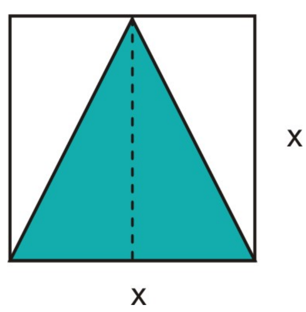
To fence the garden the way Suzie wants, we need 2 lengths and 4 widths $= 2(20) + 4(10) = 80\ feet$ of fence.

Answer: The fence is 80 feet.

Check $10 \times 20 = 200\ ft^2$ and $2(20) + 4(10) = 80\ feet$. **The answer checks out.**

Example 8

An isosceles triangle is enclosed in a square so that its base coincides with one of the sides of the square and the tip of the triangle touches the opposite side of the square. If the area of the triangle is $20\ in^2$ *what is the area of the square?*



Solution:

Draw a sketch.

Define

Let x = base of the triangle

x = height of the triangle

Translate

Area of a triangle is $\frac{1}{2} \times \text{base} \times \text{height}$, so

$$\frac{1}{2} \cdot x \cdot x = 20$$

Solve

$$ath = \frac{1}{2}x^2 = 20$$

Solve by taking the square root.

$$\begin{aligned} ath &= x^2 = 40 \\ x &= \sqrt{40} = 2\sqrt{10} \text{ and } x = -\sqrt{40} = -2\sqrt{10} \\ x &\approx 6.32 \text{ and } x \approx -6.32 \end{aligned}$$

The side of the square is 6.32 inches.

The area of the square is $(6.32)^2 = 40\ in^2$, twice as big as the area of the triangle.

Answer: Area of the triangle is 40 in^2

Check: It makes sense that the area of the square will be twice that of the triangle. If you look at the figure you can see that you can fit two triangles inside the square.

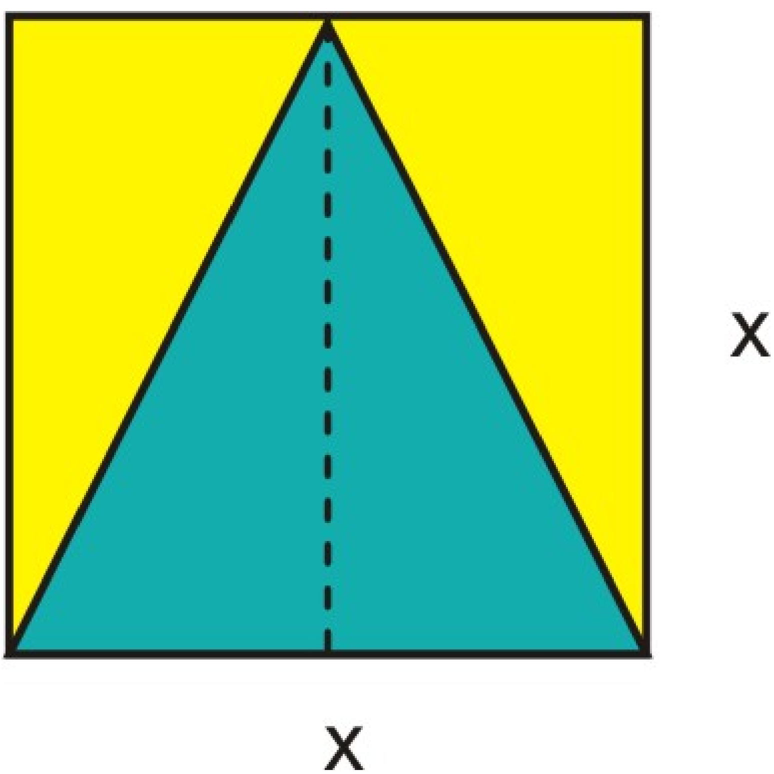


FIGURE 3.14

he answer checks out.

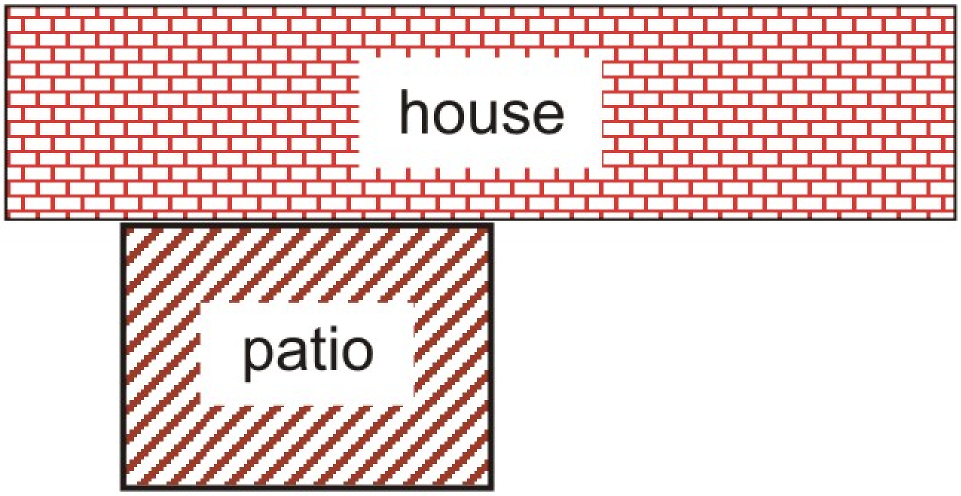
Review Questions

Solve the following quadratic equations using the quadratic formula.

1. $x^2+4x-21=0$
2. $x^2-6x=12$
3. $3x^2-\frac{1}{2}x=\frac{3}{8}$
4. $2x^2+x-3=0$
5. $-x^2-7x+12=0$
6. $-3x^2+5x=0$
7. $4x^2=0$
8. $x^2+2x+6=0$

Solve the following quadratic equations using the method of your choice.

9. $x^2-x=6$
10. $x^2-12=0$
11. $-2x^2+5x-3=0$
12. $x^2+7x-18=0$
13. $3x^2+6x=-10$
14. $-4x^2+4000x=0$
15. $-3x^2+12x+1=0$
16. $x^2+6x+9=0$
17. $81x^2+1=0$
18. $-4x^2+4x=9$
19. $36x^2-21=0$
20. $x^2-2x-3=0$
21. The product of two consecutive integers is 72. Find the two numbers.
22. The product of two consecutive odd integers is 1 less than 3 times their sum. Find the integers.
23. The length of a rectangle exceeds its width by 3 inches. The area of the rectangle is 70 square inches, find its dimensions.
24. Angel wants to cut off a square piece from the corner of a rectangular piece of plywood. The larger piece of wood is 4 $feet \times 8$ $feet$ and the cut off part is $\frac{1}{3}$ of the total area of the plywood sheet. What is the length of the side of the square?



25. Mike wants to fence three sides of a rectangular patio that is adjacent the back of his house. The area of the patio is 192 ft^2 and the length is 4 feet longer than the width. Find how much fencing Mike will need.

Review Answers

1. $x = -7, x = 3$
2. $x = 3 \pm \sqrt{21}$
3. $x = \frac{1}{12} \pm \frac{\sqrt{19}}{12}$
4. $x = -1.5, x = 1$
5. $x = \frac{7}{2} \pm \frac{\sqrt{97}}{2}$
6. $x = 1, x = \frac{2}{3}$
7. $x = 0, x = \frac{1}{4}$
8. $-1 \pm i\sqrt{5}$
9. $x = -2, x = 3$
10. $x = \pm 2\sqrt{3}$
11. $x = 1, x = 1.5$
12. $x = -9, x = 2$
13. $x = -1 \pm \frac{i\sqrt{21}}{3}$
14. $x = 0, x = 1000$
15. $x = 2 \pm \frac{\sqrt{39}}{3}$
16. $x = -3$
17. $x = \pm \frac{1}{9}i$
18. $\frac{1}{2} \pm i\sqrt{2}$
19. $x = \pm \frac{\sqrt{21}}{6}$
20. $x = -1, x = 3$
21. 8 and 9
22. 5 and 7
23. 7 in and 10 in
24. side $= 3.27\ ft$
25. 40 feet of fencing.

3.6 The Discriminant

Learning Objectives

- Find the discriminant of a quadratic equation.
- Interpret the discriminant of a quadratic equation.
- Solve real-world problems using quadratic functions and interpreting the discriminant.

Introduction

The quadratic equation is $ax^2 + bx + c = 0$.

It can be solved using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The expression inside the square root is called the **discriminant**, $D = b^2 - 4ac$. The discriminant can be used to analyze the types of solutions of quadratic equations without actually solving the equation. Here are some guidelines.

- If $b^2 - 4ac > 0$, we obtain two separate real solutions.
- If $b^2 - 4ac < 0$, we obtain non-real solutions.
- If $b^2 - 4ac = 0$, we obtain one real solution, a **double root.**

Find the Discriminant of a Quadratic Equation

To find the discriminant of a quadratic equation, we calculate $D = b^2 - 4ac$.

Example 1

Find the discriminant of each quadratic equation. Then tell how many solutions there will be to the quadratic equation without solving.

a) $x^2 - 5x + 3 = 0$

b) $4x^2 - 4x + 1 = 0$

c) $-2x^2 + x = 4$

Solution:

a) Substitute $a = 1, b = -5$ and $c = 3$ into the discriminant formula $D = (-5)^2 - 4(1)(3) = 13$.

There are two real solutions because $D > 0$.

b) Substitute $a = 4, b = -4$ and $c = 1$ into the discriminant formula $D = (-4)^2 - 4(4)(1) = 0$.

There is one real solution because $D = 0$.

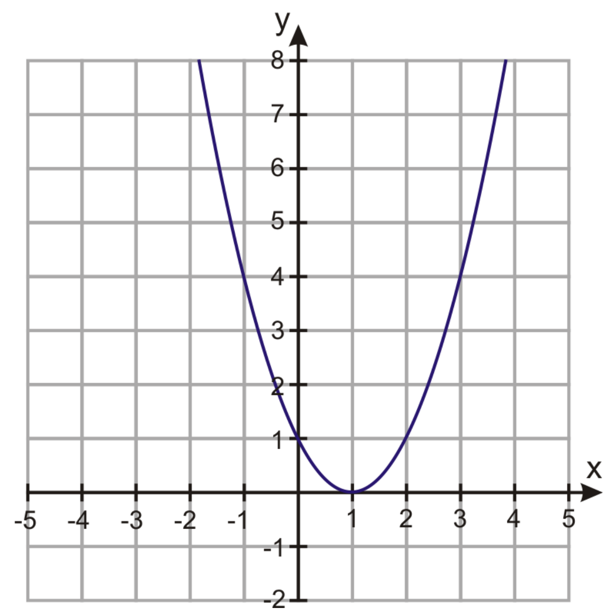
c) Rewrite the equation in standard form $-2x^2 + x - 4 = 0$.

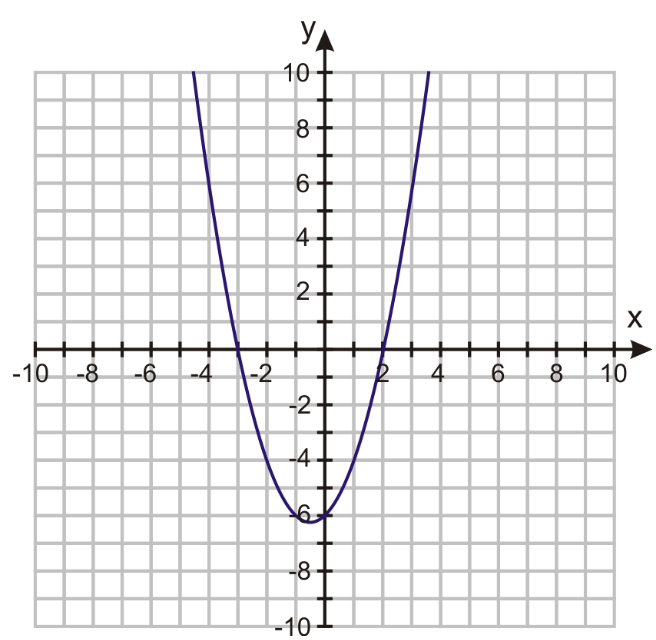
Substitute $a = -2, b = 1$ and $c = -4$ into the discriminant formula: $D = (1)^2 - 4(-2)(-4) = -31$.

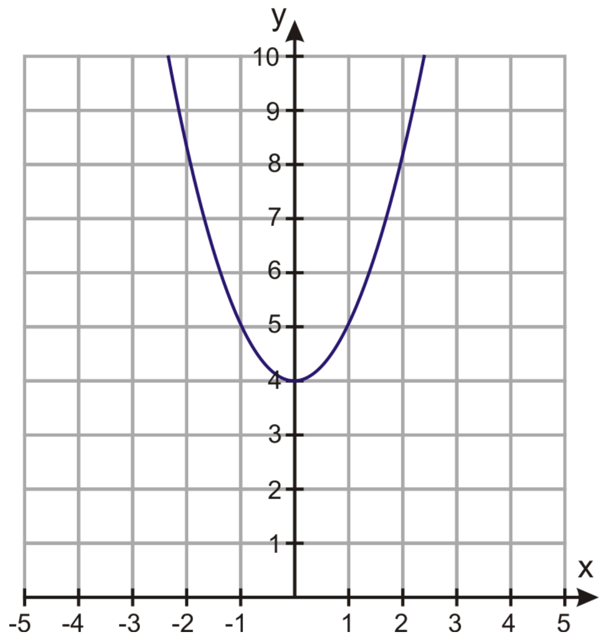
There are no real solutions because $D < 0$.

Interpret the Discriminant of a Quadratic Equation

The sign of the discriminant tells us the nature of the solutions (or roots) of a quadratic equation. We can obtain two distinct real solutions if $D > 0$, no real solutions if $D < 0$ or one solution (called a "double root") if $D = 0$. Recall that the number of solutions of a quadratic equation tell us how many times a parabola crosses the $x-$axis.







Example 2

Determine the nature of solutions of each quadratic equation.

a) $4x^2 - 1 = 0$

b) $10x^2 - 3x = -4$

c) $x^2 - 10x + 25 = 0$

Solution

Use the value of the discriminant to determine the nature of the solutions to the quadratic equation.

a) Substitute $a = 4, b = 0$ and $c = -1$ into the discriminant formula $D = (0)^2 - 4(4)(-1) = 16$.

The discriminant is positive, so the equation has two distinct real solutions.

The solutions to the equation are: $\frac{0 \pm \sqrt{16}}{8} = \pm \frac{4}{8} = \pm \frac{1}{2}$.

b) Rewrite the equation in standard form $10x^2 - 3x + 4 = 0$.

Substitute $a = 10, b = -3$ and $c = 4$ into the discriminant formula $D = (-3)^2 - 4(10)(4) = -151$.

The discriminant is negative, so the equation has two non-real solutions.

c) Substitute $a = 1, b = -10$ and $c = 25$ into the discriminant formula $D = (-10)^2 - 4(1)(25) = 0$.

The discriminant is 0, so the equation has a double root.

The solution to the equation is $\frac{10 \pm \sqrt{0}}{2} = \frac{10}{2} = 5$.

If the discriminant is a perfect square, then the solutions to the equation are rational numbers.

Example 3

Determine the nature of the solutions to each quadratic equation.

a) $2x^2 + x - 3 = 0$

b) $-x^2 - 5x + 14 = 0$

Solution

Use the discriminant to determine the nature of the solutions.

a) Substitute $a = 2, b = 1$ and $c = -3$ into the discriminant formula $D = (1)^2 - 4(2)(-3) = 25$.

The discriminant is a positive perfect square so the solutions are two real rational numbers.

The solutions to the equation are $\frac{-1 \pm \sqrt{25}}{4} = \frac{-1 \pm 5}{4}$ so, $x = 1$ and $x = -\frac{3}{2}$.

b) Substitute $a = -1, b = -5$ and $c = 14$ into the discriminant formula: $D = (-5)^2 - 4(-1)(14) = 81$.

The discriminant is a positive perfect square so the solutions are two real rational numbers.

The solutions to the equation are $\frac{5 \pm \sqrt{81}}{-2} = \frac{5 \pm 9}{-2}$ so, $x = -7$ and $x = 2$.

If the discriminant is not a perfect square, then the solutions to the equation are irrational numbers.

Example 4

Determine the nature of the solutions to each quadratic equation.

a) $-3x^2 + 4x + 1 = 0$

b) $5x^2 - x - 1 = 0$

Solution

Use the discriminant to determine the nature of the solutions.

a) Substitute $a = -3, b = 2$ and $c = 1$ into the discriminant formula $D = (4)^2 - 4(-3)(1) = 28$.

The discriminant is a positive perfect square, so the solutions are two real irrational numbers.

The solutions to the equation are $\frac{-2 \pm \sqrt{28}}{-6}$ so, $x \approx -0.55$ and $x \approx 1.22$.

b) Substitute $a = 5, b = -1$ and $c = -1$ into the discriminant formula $D = (-1)^2 - 4(5)(-1) = 21$.

The discriminant is a positive perfect square so the solutions are two real irrational numbers.

The solutions to the equation are $\frac{1 \pm \sqrt{20}}{10}$ so, $x \approx 0.56$ and $x \approx -0.36$.

Solve Real-World Problems Using Quadratic Functions and Interpreting the Discriminant

You saw that calculating the discriminant shows what types of solutions a quadratic equation possesses. Knowing the types of solutions is very useful in applied problems. Consider the following situation.

Example 5

Marcus kicks a football in order to score a field goal. The height of the ball is given by the equation $y = -\frac{32}{6400}x^2 + x$ *where y is the height and x is the horizontal distance the ball travels. We want to know if he kicked the ball hard enough to go over the goal post which is 10 feet high.*

Solution

Define

Let y = height of the ball in feet

x = distance from the ball to the goalpost.

Translate We want to know if it is possible for the height of the ball to equal 10 feet at some real distance from the goalpost.

$$10 = -\frac{32}{6400}x^2 + x$$

Solve

Write the equation in standard form.	$-\frac{32}{6400}x^2 + x - 10 = 0$
Simplify.	$-0.005x^2 + x - 10 = 0$
Find the discriminant.	$D = (1)^2 - 4(-0.005)(-10) = 0.8$

Since the discriminant is positive, we know that it is possible for the ball to go over the goal post, if Marcus kicks it from an acceptable distance x from the goal post. From what distance can he score a field goal? See the next example.

Example 6 (continuation)

What is the farthest distance that he can kick the ball from and still make it over the goal post?

Solution

We need to solve for the value of x by using the quadratic formula.

$$x = \frac{-1 \pm \sqrt{0.8}}{-0.01} \approx 10.6 \text{ or } 189.4$$

This means that Marcus has to be closer that 189.4 feet or further than 10.6 feet to make the goal. (Why are there two solutions to this equation? Think about the path of a ball after it is kicked).

Example 7

Emma and Bradon own a factory that produces bike helmets. Their accountant says that their profit per year is given by the function

$$P = 0.003x^2 + 12x + 27760$$

In this equation x is the number of helmets produced. Their goal is to make a profit of $40,000 this year. Is this possible?

Solution

We want to know if it is possible for the profit to equal $40,000.

$$40000 = -0.003x^2 + 12x + 27760$$

Solve

Write the equation in standard form	$-0.003x^2 + 12x - 12240 = 0$
Find the discriminant.	$D = (12)^2 - 4(-0.003)(-12240) = -2.88$

Since the discriminant is negative, we know that there are no real solutions to this equation. Thus, it is not possible for Emma and Bradon to make a profit of $40,000 this year no matter how many helmets they make.

Review Questions

Find the discriminant of each quadratic equation.

1. $2x^2 - 4x + 5 = 0$
2. $x^2 - 5x = 8$
3. $4x^2 - 12x + 9 = 0$
4. $x^2 + 3x + 2 = 0$
5. $x^2 - 16x = 32$
6. $-5x^2 + 5x - 6 = 0$

Determine the nature of the solutions of each quadratic equation.

7. $-x^2 + 3x - 6 = 0$
8. $5x^2 = 6x$
9. $41x^2 - 31x - 52 = 0$
10. $x^2 - 8x + 16 = 0$
11. $-x^2 + 3x - 10 = 0$
12. $x^2 - 64 = 0$

Without solving the equation, determine whether the solutions will be rational or irrational.

13. $x^2 = -4x + 20$
14. $x^2 + 2x - 3 = 0$
15. $3x^2 - 11x = 10$
16. $\frac{1}{2}x^2 + 2x + \frac{2}{3} = 0$
17. $x^2 - 10x + 25 = 0$
18. $x^2 = 5x$
19. Marty is outside his apartment building. He needs to give Yolanda her cell phone but he does not have time to run upstairs to the third floor to give it to her. He throws it straight up with a vertical velocity of 55 feet/second. Will the phone reach her if she is 36 feet up?
(Hint: The equation for the height is given by $y = -32t^2 + 55t + 4$.)
20. Bryson owns a business that manufactures and sells tires. The revenue from selling the tires in the month of July is given by the function $R = x(200 - 0.4x)$ where x is the number of tires sold. Can Bryson's business generate revenue of $20,000 in the month of July?

Review Answers

1. $D = -24$
2. $D = 57$
3. $D = 0$
4. $D = 1$
5. $D = 384$
6. $D = -95$
7. $D = -15$ no real solutions
8. $D = 36$ two real solutions
9. $D = 9489$ two real solutions
10. $D = 0$ one real solutions
11. $D = -31$ no real solutions
12. $D = 256$ two real solutions
13. $D = 96$ two real irrational solutions

14. $D = 16$ two real rational solutions
15. $D = 241$ two real irrational solutions
16. $D = \frac{8}{3}$ two real irrational solutions
17. $D = 0$ one real rational solution
18. $D = 25$ two real rational solutions
19. no
20. yes

3.7 Linear and Quadratic Models

Learning Objectives

- Identify functions using differences and ratios.
- Write equations for functions.
- Perform exponential and quadratic regressions with a graphing calculator.
- Solve real-world problems by comparing function models.

Introduction

In this course you have learned about three types of functions, linear, quadratic and exponential.

Linear functions take the form $y = mx = b$ or $f(x) = mx + b$.

Quadratic functions take the form $y = ax^2 + bx + c$ or $f(x) = ax^2 + bx + c$.

In real-world applications, the function that describes some physical situation is not given. Finding the function is an important part of solving problems. For example, scientific data such as observations of planetary motion are often collected as a set of measurements given in a table. One job for the scientist is to figure out which function best fits the data. In this section, you will learn some methods that are used to identify which function describes the relationship between the dependent and independent variables in a problem.

Identify Functions Using Differences or Ratios.

One method for identifying functions is to look at the difference or the ratio of different values of the dependent variable.

We use differences to identify linear functions.

If the difference between values of the dependent variable is the same each time we change the independent variable by the same amount, then the function is *linear.*

Example 1

Determine if the function represented by the following table of values is linear.

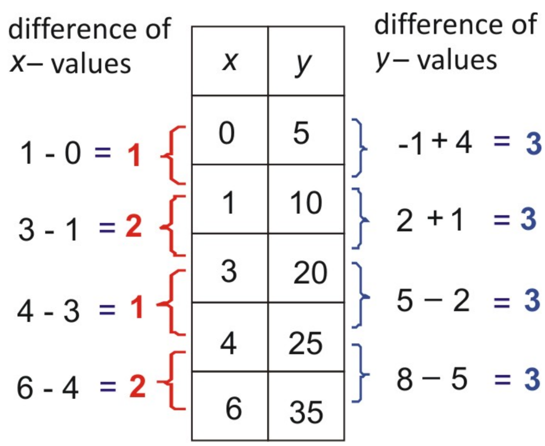
x	y	difference of $y-$ values
-2	-4	
		-1 + 4 = 3
-1	-1	
		2 + 1 = 3
0	2	
		5 − 2 = 3
1	5	
		8 − 5 = 3
2	8	

If we take the difference between consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value always increases by 3.

Since the difference is always the same, **the function is linear.**

When we look at the difference of the $y-$values, we must make sure that we examine entries for which the $x-$values increase by the same amount.

For example, examine the values in the following table.



At first glance, this function might not look linear because the difference in the $y-$values is not always the same.

However, we see that the difference in $y-$values is 5 when we increase the $x-$values by 1, and it is 10 when we increase the $x-$values by 2. This means that the difference in $y-$values is always 5 when we increase the $x-$values by 1. Therefore, the function is linear. The key to this observation is that **the ratio of the differences is constant**. This function is $f(x) = 5x + 5$.

In mathematical notation, we can write the linear property as follows.

If $\frac{y_2 - y_1}{x_2 - x_1}$ is always the same for values of the dependent and independent variables, then the points are on a line. Notice that the expression we wrote is the definition of the slope of a line.

Differences can also be used to identify quadratic functions. For a quadratic function, when we increase the $x-$values by the same amount,

the difference between $y-$values will not be the same. However, the difference of the differences of the $y-$values will be the same.

Here are some examples of quadratic relationships represented by tables of values.

a)

x	$y = x^2$	difference of $y-$ values	difference of differences
0	0	$1 - 0 = 1$	
1	1	$4 - 1 = 3$	$3 - 1 = 2$
2	4	$9 - 4 = 5$	$5 - 3 = 2$
3	9	$16 - 9 = 7$	$7 - 5 = 2$
4	16	$25 - 16 = 9$	$9 - 7 = 2$
5	25	$36 - 25 = 11$	$11 - 9 = 2$
6	36		

In this quadratic function, $f(x) = x^2$, when we increase the $x-$value by one, the value of y increases by different values. However, the increase is constant: the difference of the difference is always 2.

b)

x	$y = 2x^2\ 3x + 1$	difference of $y-$ values	difference of differences
0	0	$0 - 1 = -1$	
1	1	$3 - 0 = 3$	$3 + 1 = 4$
2	3	$10 - 3 = 7$	$7 - 3 = 4$
3	10	$21 - 10 = 11$	$11 - 7 = 4$
4	21	$36 - 21 = 15$	$15 - 11 = 4$
5	36	$55 - 36 = 19$	$19 - 15 = 4$
6	55		

In this quadratic function, $f(x) = x^2 - 3x + 1$, when we increase the $x-$value by one, the value of y increases by different values. However, the increase is constant: the difference of the difference is always 4.

We use ratios to identify exponential functions.

Write Equations for Functions.

Once we identify which type of function fits the given values, we can write an equation for the function by starting with the general form for that type of function.

Example 2

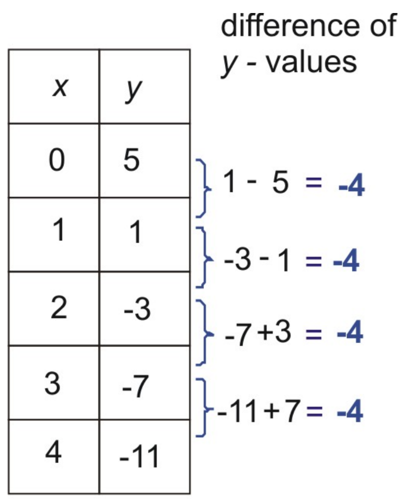
Determine what type of function represents the values in the following table.

TABLE 3.16:

x	y
0	3
1	1
2	-3
3	-7
4	-11

Solution

Let's first check the difference of consecutive values of y.



If we take the difference between consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value always decreases by 4. Since the difference is always the same, **the function is linear.**

To find the equation for the function that represents these values, we start with the general form of a linear function.

$$f(x) = mx + b$$

Here m is the slope of the line and is defined as the quantity by which y increases every time the value of x increases by one. The constant b is the value of the function when $x = 0$. Therefore, the function is

$$f(x) = -4x + 5$$

Example 3

Determine what type of function represents the values in the following table.

TABLE 3.17:

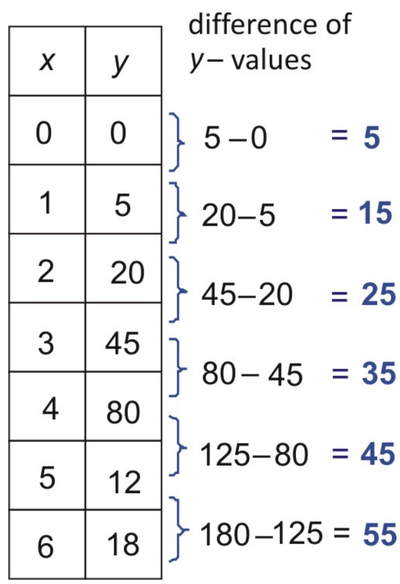
x	y
0	0
1	5

TABLE 3.17: (continued)

x	y
2	20
3	45
4	80
5	125
6	180

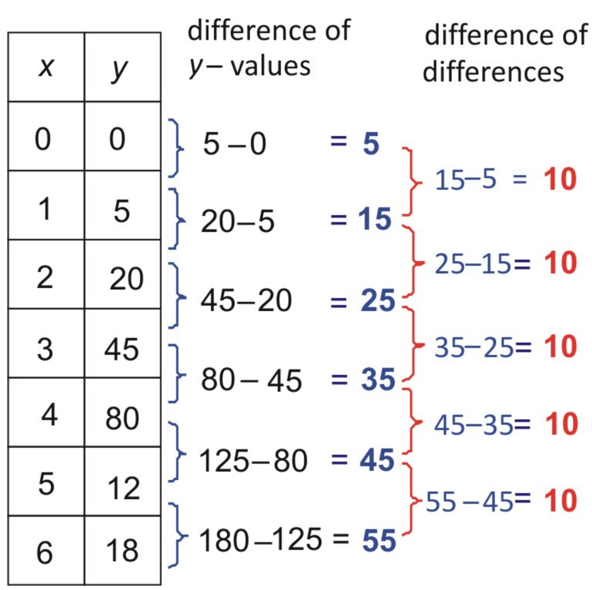
Solution

Let's first check the difference of consecutive values of y.



If we take the difference between consecutive $y-$values, we see that each time the $x-$value increases by one, the $y-$value does not remain constant. Since the difference is not the same, **the function is not linear.**

Now, let's check the difference of the differences in the values of y.



When we increase the $x-$value by one, the value of y increases by different values. However, the increase is constant. The difference of the differences is always 10 when we increase the $x-$value by one.

The function describing these set of values is **quadratic.** To find the equation for the function that represents these values, we start with the general form of a quadratic function.

$$f(x) = ax^2 + bx + c$$

We need to use the values in the table to find the values of the constants a, b and c.

The value of c represents the value of the function when $x = 0$, so $c = 0$.

$$\begin{aligned}
\text{Then} \qquad & y = ax^2 + bx \\
\text{Plug in the point } (1,5). \qquad & 5 = a + b \\
\text{Plug in the point } (2,20). \qquad & 20 = 4a + 2b \Rightarrow 10 = 2a + b \\
\text{To find } a \text{ and } b, \text{ we solve the system of equations} \qquad & 5 = a + b \\
& 10 = 2a + b \\
\text{Solve the first equation for } b. \qquad & 5 = a + b \Rightarrow b = 5 - a \\
\text{Plug the first equation into the second.} \qquad & 10 = 2a + 5 - a \\
\text{Solve for a and } b. \qquad & a = 5 \text{ and } b = 0
\end{aligned}$$

Therefore the equation of the quadratic function is

$$y = 5x^2$$

Perform Quadratic Regressions with a Graphing Calculator.

Earlier you learned how to perform linear regression with a graphing calculator to find the equation of a straight line that fits a linear data set. In this section, you will learn how to perform exponential and quadratic regression to find equations for functions that describe non-linear relationships between the variables in a problem.

Example 4

Find the quadratic function that is a best fit for the data in the following table. The following table shows how many miles per gallon a car gets at different speeds.

TABLE 3.18:

Speed (mi/h)	Miles Per Gallon
30	18
35	20
40	23
45	25
50	28
55	30
60	29
65	25
70	25

Using a graphing calculator.

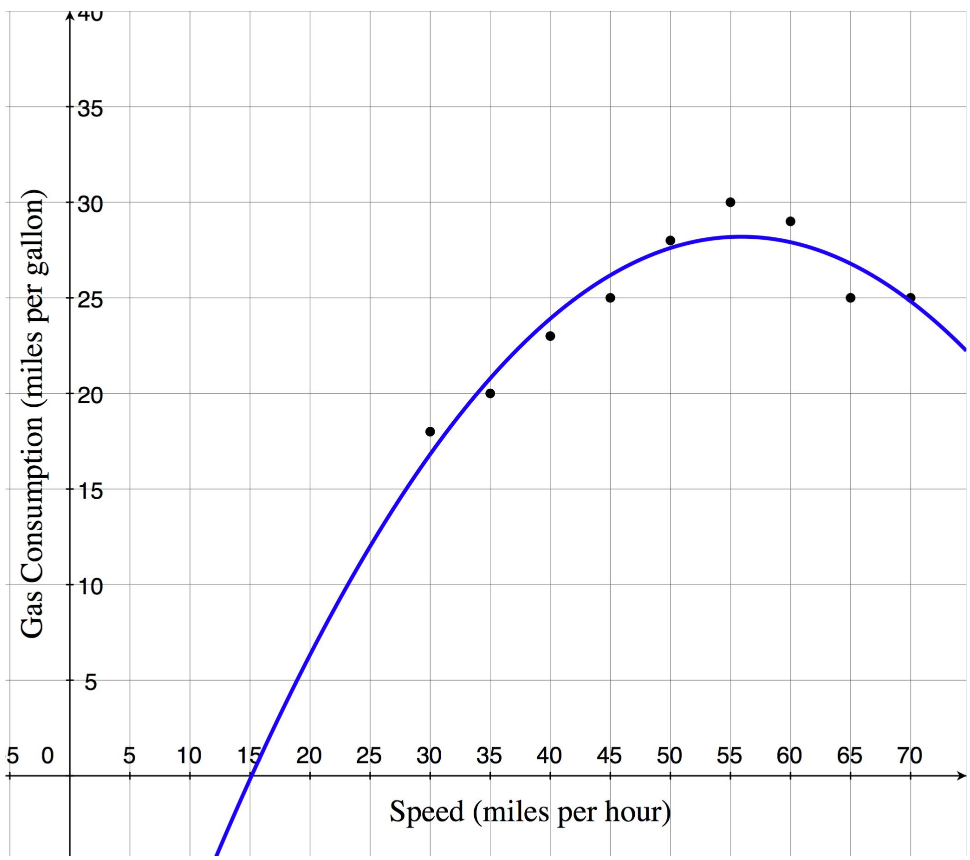
a) Draw the scatterplot of the data.

b) Find the quadratic function of best fit.

c) Draw the quadratic function of best fit on the scatterplot.

d) Find the speed that maximizes the miles per gallon.

e) Predict the miles per gallon of the car if you drive at a speed of 48 miles per gallon.



Solution

Step 1 Input the data

Press **[STAT]** and choose the **[EDIT]**option.

Input the values of x in the first column (L_1) and the values of y in the second column (L_2).

Note: In order to clear a list, move the cursor to the top so that L_1 or L_2 is highlighted. Then press **[CLEAR]** button and then **[ENTER]**.

Step 2 Draw the scatter plot.

First press **[Y=]** and clear any function on the screen by pressing **[CLEAR]** when the old function is highlighted.

Press **[STATPLOT] [STAT]** and **[Y=]** and choose option 1.

Choose the ON option, after TYPE, choose the first graph type (scatterplot) and make sure that the Xlist and Ylist names match the names on top of the columns in the input table.

Press **[GRAPH]** and make sure that the window is set so you see all the points in the scatterplot. In this case $30 \leq x \leq 80$ and $0 \leq y \leq 40$.

You can set the window size by pressing on the **[WINDOW]** key at top.

Step 3 Perform quadratic regression.

Press **[STAT]** and use right arrow to choose **[CALC]**.

Choose Option 5 (QuadReg) and press **[ENTER]**. You will see "QuadReg" on the screen.

Type in L_1, L_2 after 'QuadReg' and Press **[ENTER]**. The calculator shows the quadratic function.

Function $f(x) = -0.017x^2 + 1.9x - 25$

Step 4: Graph the function.

Press **[Y=]** and input the function you just found.

Press **[GRAPH]** and you will see the curve fit drawn over the data points.

To find the speed that maximizes the miles per gallons, use **[TRACE]** and move the cursor to the top of the parabola. You can also use **[CALC] [2nd] [TRACE]** and option 4 Maximum, for a more accurate answer. The speed that maximizes miles per gallons $= 56\ mi/h$

Plug $x = 56$ into the equation you found: $f(56) = -0.017(56)^2 = 1.9(56) - 25 = 28$ miles per gallon

Note: The image to the right shows our data points from the table and the function plotted on the same graph. One thing that is clear from this graph is that predictions made with this function will not make sense for all values of x. For example, if $x < 15$, this graph predicts that we will get negative mileage, something that is impossible. Thus, part of the skill of using regression on your calculator is being aware of the strengths and limitations of this method of fitting functions to data.

Solve Real-World Problems by Comparing Function Models

Example 7

The following table shows the number of students enrolled in public elementary schools in the US (source: US Census Bureau). Make a scatterplot with the number of students as the dependent variable, and the number of years since 1990 as the independent variable. Find which curve fits this data the best and predict the school enrollment in the year 2007.

TABLE 3.19:

Year	Number of Students (millions)
1990	26.6
1991	26.6
1992	27.1
1993	27.7
1994	28.1
1995	28.4
1996	28.1
1997	29.1
1998	29.3
2003	32.5

Solution

We will perform linear and quadratic regression on this data set and see which function represents the values in the table the best.

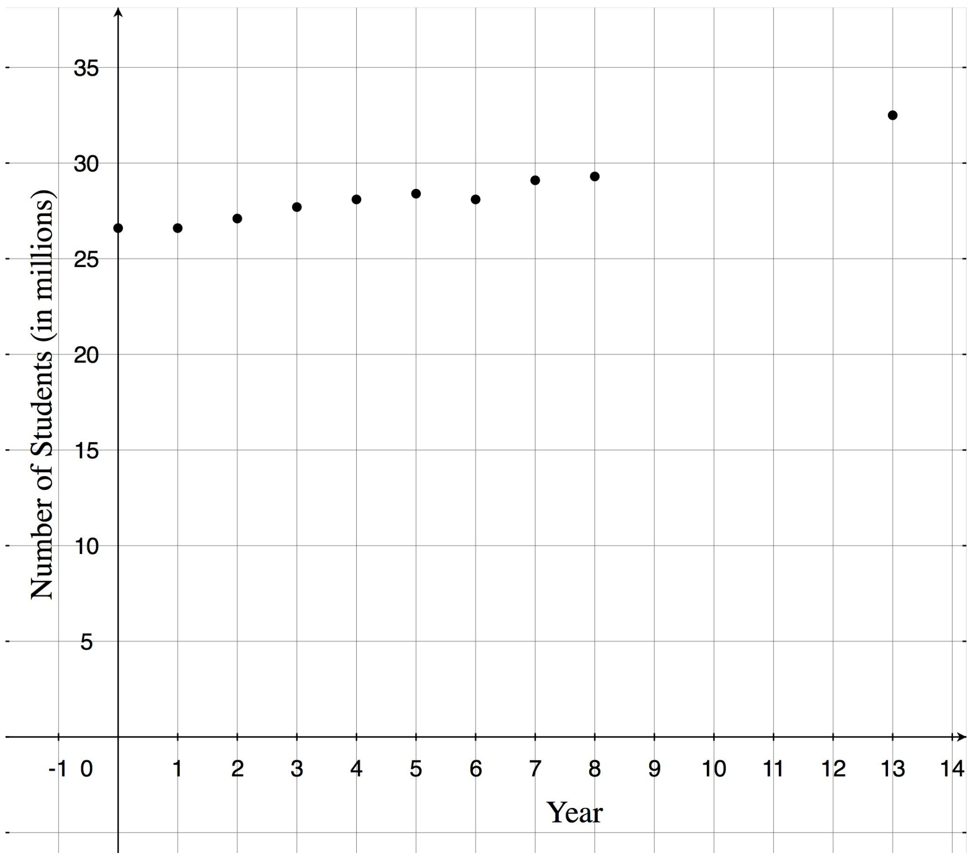
Step 1 Input the data.

Input the values of x in the first column (L_1) and the values of y in the second column (L_2).

Step 2 Draw the scatter plot.

Set the window size: $0 \leq x \leq 10$ and $20 \leq y \leq 40$.

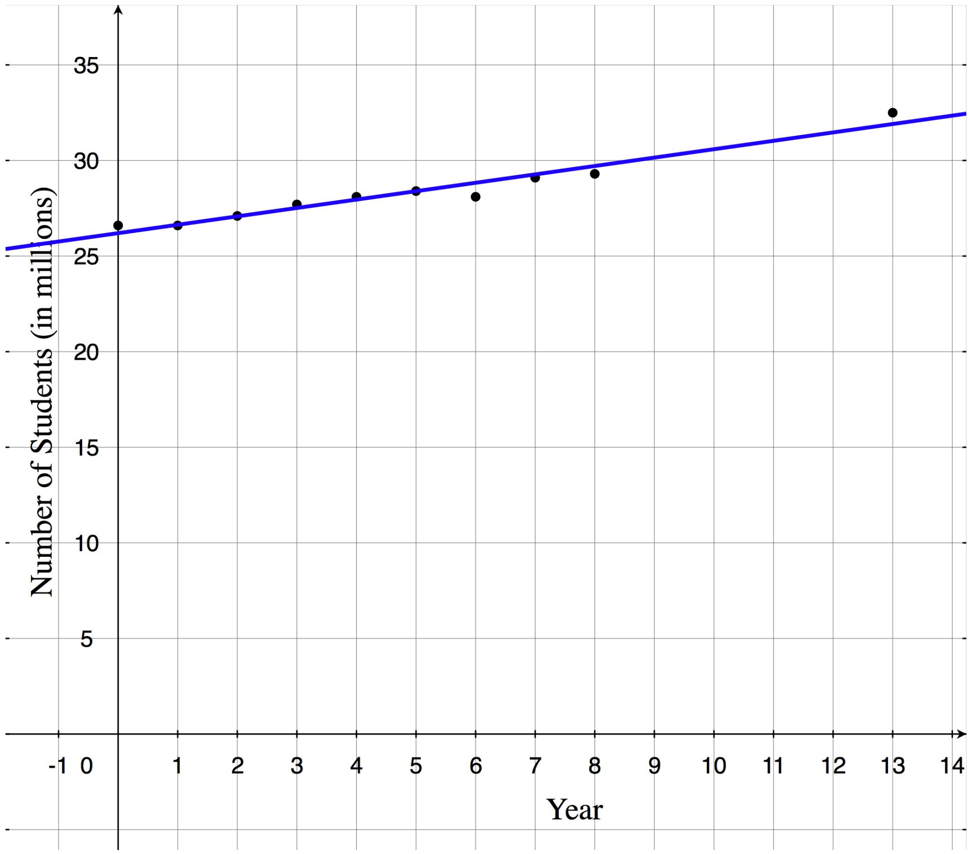
Here is the scatter plot.



Step 3 Perform Regression.

Linear Regression

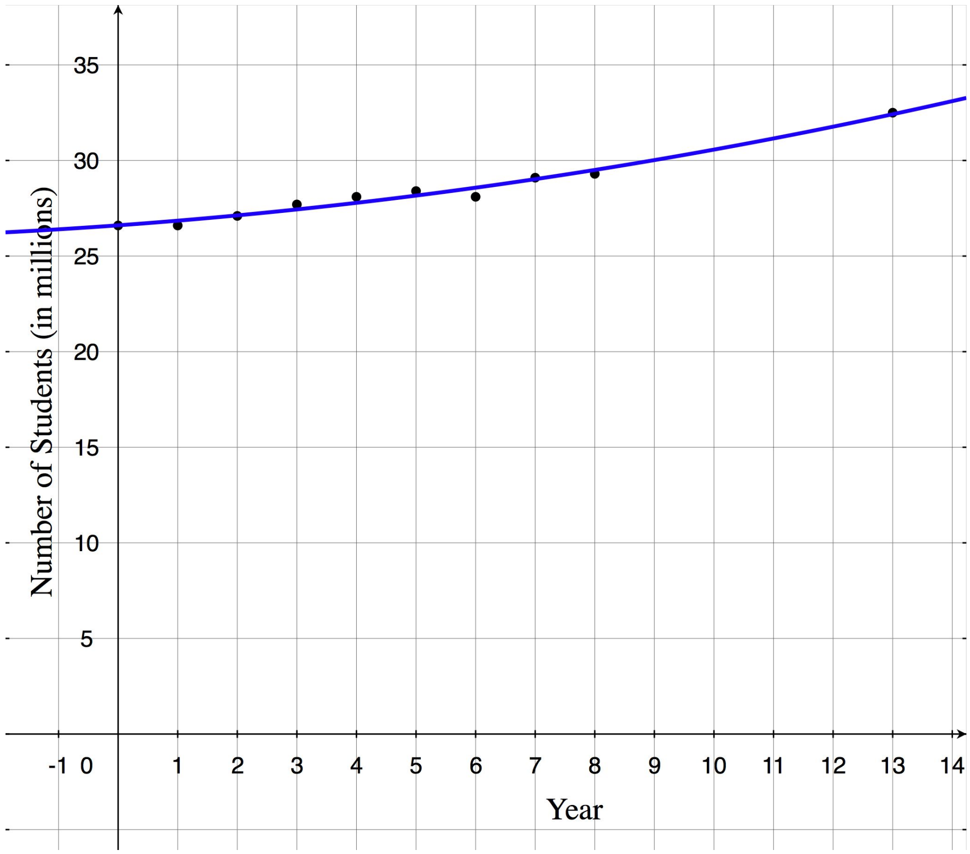
The function of the line of best fit is $f(x) = 0.44x + 26.1$.



Here is the graph of the function on the scatter plot.

Quadratic Regression

The quadratic function of best fit is $f(x) = 0.064x^2 - .067x + 26.84$.



Here is the graph of the function on the scatter plot.

Review Questions

Determine whether the data in the following tables can be represented by a linear function.

TABLE 3.20:

x	y
-4	10
-3	7
-2	4
-1	1
0	-2
1	-5

TABLE 3.21:

x	y
-2	4
-1	3
0	2
1	3
2	6
3	11

TABLE 3.22:

x	y
0	50
1	75
2	100
3	125
4	150
5	175

Determine whether the data in the following tables can be represented by a quadratic function:

TABLE 3.23:

x	y
-10	10
-5	2.5
0	0
5	2.5
10	10
15	22.5

TABLE 3.24:

x	y
1	4
2	6
3	6
4	4
5	0
6	-6

TABLE 3.25:

x	y
-3	-27
-2	-8
-1	-1
0	0
1	1
2	8
3	27

Determine what type of function represents the values in the following table and find the equation of the function.

TABLE 3.26:

x	y
-9	-3
-7	-2
-5	-1

TABLE 3.26: (continued)

x	y
-3	0
-1	1
1	2

TABLE 3.27:

x	y
-3	14
-2	4
-1	-2
0	-4
1	-2
2	4
3	14

9. As a ball bounces up and down, the maximum height that the ball reaches continually decreases from one bounce to the next. For a given bounce, the table shows the height of the ball with respect to time.

TABLE 3.28:

Time (seconds)	**Height (inches)**
2	2
2.2	16
2.4	24
2.6	33
2.8	38
3.0	42
3.2	36
3.4	30
3.6	28
3.8	14
4.0	6

Using a graphing calculator

1. Draw the scatter plot of the data.
2. Find the quadratic function of best fit.
3. Draw the quadratic function of best fit on the scatter plot.
4. Find the maximum height the ball reaches on the bounce.
5. Predict how high the ball is at time $t = 2.5\ seconds$.

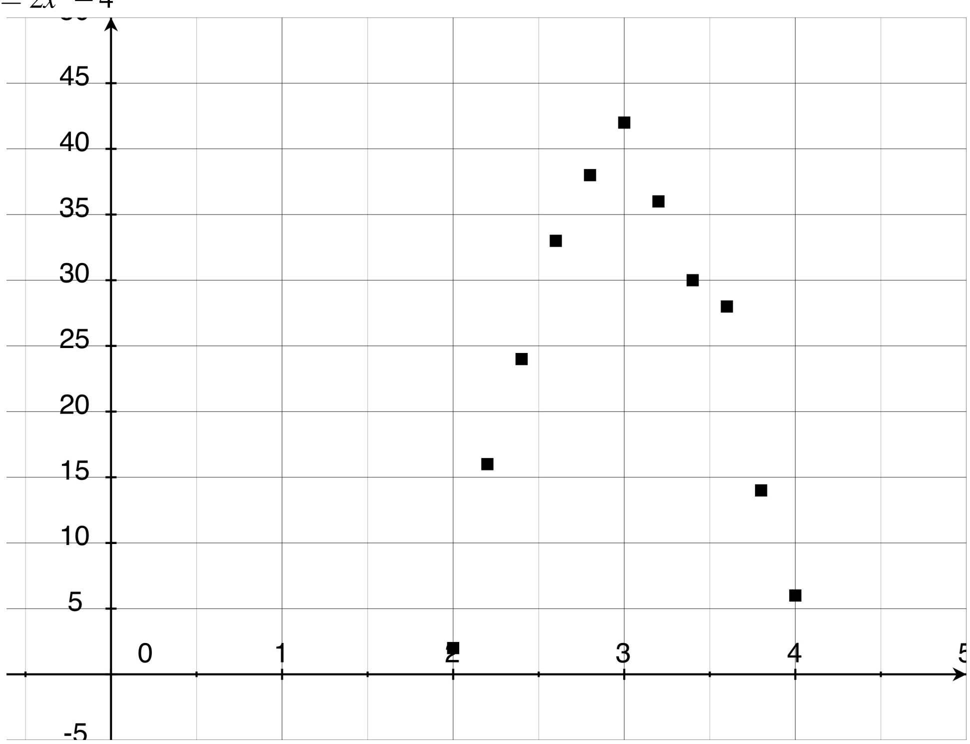
10. The following table shows the rate of pregnancies (per 1000) for US women aged 15 to 19. (source: US Census Bureau). Make a scatterplot with the rate of pregnancies as the dependent variable and the number of years since 1990 as the independent variable. Find which curve fits this data the best and predict the rate of teen pregnancies in the year 2010.

TABLE 3.29:

Year	Rate of Pregnancy (per 1000)
1990	116.9
1991	115.3
1992	111.0
1993	108.0
1994	104.6
1995	99.6
1996	95.6
1997	91.4
1998	88.7
1999	85.7
2000	83.6
2001	79.5
2002	75.4

Review Answers

1. Linear common difference $= -3$
2. Not Linear
3. Linear common difference $= 25$
4. Quadratic difference of difference $= 5$
5. Quadratic difference of difference $= -2$
6. Not Quadratic
7. Linear $f(x) = \left(\frac{1}{2}\right)x + \left(\frac{3}{2}\right)$
8. Quadratic $f(x) = 2x^2 - 4$



9. (a)
   (b) $f(x) = -35.4x^2 + 213.3x - 282.4$;
   (c) Maximum height $= 38.9$ *inches*.
   (d) $t = 2.5$ *sec*, height $= 29.6$ *inches*.
10. linear function is best fit: $f(x) = -3.54x + 117.8$ In year 2010, $x = 20$, rate of teen pregnancies $= 47$ per 1000

3.8 Problem Solving Strategies: Choose a Function Model

Learning Objectives

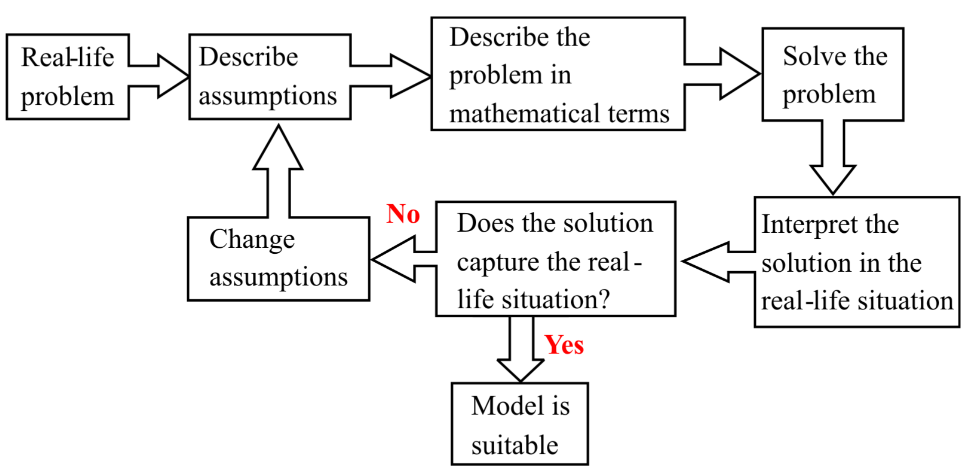
- Read and understand given problem situations
- Develop and use the strategy: Choose a Function
- Develop and use the strategy: Make a Model
- Plan and compare alternative approaches to solving problems
- Solve real-world problems using selected strategies as part of a plan

Introduction

As you learn more and more mathematical methods and skills, it is important to think about the purpose of mathematics and how it works as part of a bigger picture. Mathematics is used to solve problems which often arise from real-life situations. **Mathematical modeling** is a process by which we start with a real-life situation and arrive at a quantitative solution. Modeling involves creating a set of mathematical equations that describes a situation, solving those equations and using them to understand the real-life problem. Often the model needs to be adjusted because it does not describe the situation as well as we wish.

A mathematical model can be used to gain understanding of a real-life situation by learning how the system works, which variables are important in the system and how they are related to each other. Models can also be used to predict and forecast what a system will do in the future or for different values of a parameter. Lastly, a model can be used to estimate quantities that are difficult to evaluate exactly.

Mathematical models are like other types of models. The goal is not to produce an exact copy of the "real" object but rather to give a representation of some aspect of the real thing. The modeling process can be summarized as follows.



Notice that the modeling process is very similar to the problem solving format we have been using throughout this book. In this section, we will focus mostly on the assumptions we make and the validity of the model. Functions are an integral part of the modeling process because they are used to describe the mathematical relationship in a system. One of the most difficult parts of the modeling process is determining which function best describes a situation. We often find that the function we chose is not appropriate. Then, we must choose a different one, or we findthat a function model is good for one set of parameters but we need to use another function for a different set of parameters. Often, for certain parameters, more than one function describes the situation well and using the simplest function is most practical.

Here we present some mathematical models arising from real-world applications.

Example 1 Stretching springs beyond the "elastic limit"

A spring is stretched as you attach more weight at the bottom of the spring. The following table shows the length of the spring in inches for different weights in ounces.

Weight (oz)	0	2	4	6	8	10	12	14	16	18	20
Length (in)	2	2.4	2.8	3.2	3.5	3.9	4.1	4.4	4.6	4.7	4.8

a) Find the length of the spring as a function of the weight attached to it.

b) Find the length of the spring when you attach 5 ounces.

c) Find the length of the spring when you attach 19 ounces.

Solution

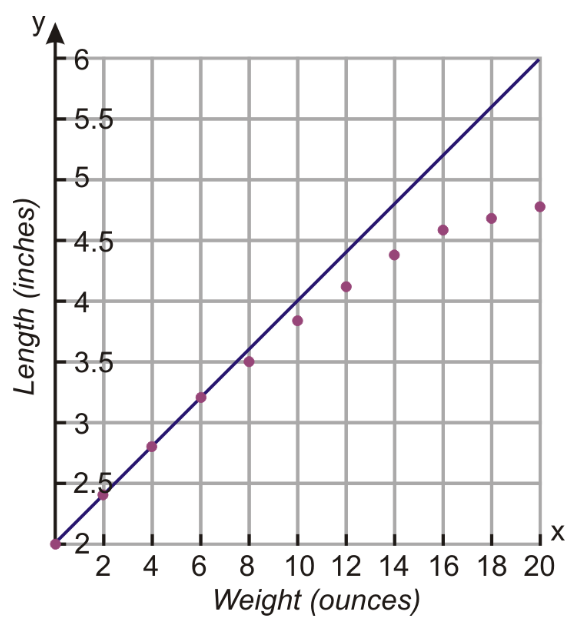
Step 1 Understand the problem

Define x = weight in ounces on the spring

y = length in inches of the spring

Step 2 Devise a plan

Springs usually have a linear relationship between the weight on the spring and the stretched length of the spring. If we make a scatter plot, we notice that for lighter weights the points do seem to fit on a straight line (see graph). Assume that the function relating the length of the spring to the weight is linear.



Step 3 Solve

Find the equation of the line using points describing lighter weights:

(0, 2) and (4, 2.8).

The slope is $m = \frac{.8}{4} = 0.2$

Using $y = mx + b$

a) We obtain the function $y = .2x + 2$.

b) To find the length of the spring when the weight is 5 ounces, we plug in $x = 5$.

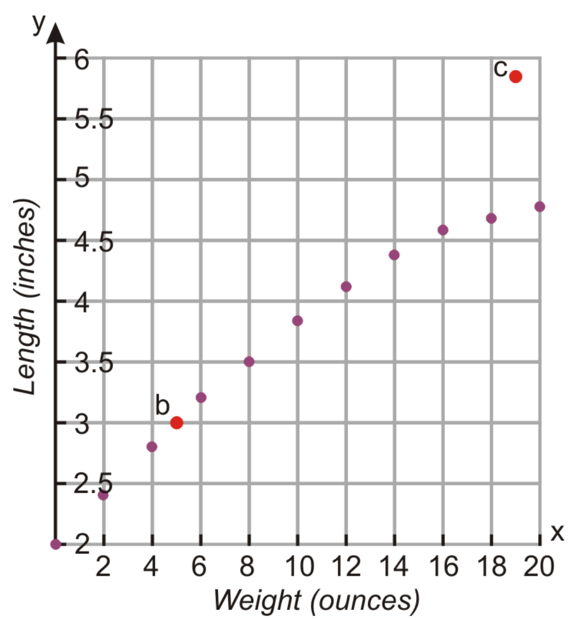
$$y = .2(5) + 2 = 3 \ inches$$

c) To find the length of the spring when the weight is 19 ounces, we plug in $x = 19$.

$$y = .2(19) + 2 = 5.8 \ inches$$

Step 4 Check

To check the validity of the solutions lets plot the answers to b) and c) on the scatter plot. We see that the answer to b) is close to the rest of the data, but the answer to c) does not seem to follow the trend.



We can conclude that for small weights, the relationship between the length of the spring and the weight is a linear function.

For larger weights, the spring does not seem to stretch as much for each added ounces. We must change our assumption. There must be a non-linear relationship between the length and the weight.

Step 5 Solve with New Assumptions

Let's find the equation of the function by cubic regression with a graphing calculator.

a) We obtain the function $y = -.000145x^3 - .000221x^2 + .202x + 2.002$.

b) To find the length of the spring when the weight is 5 ounces, we plug in $x = 5$.

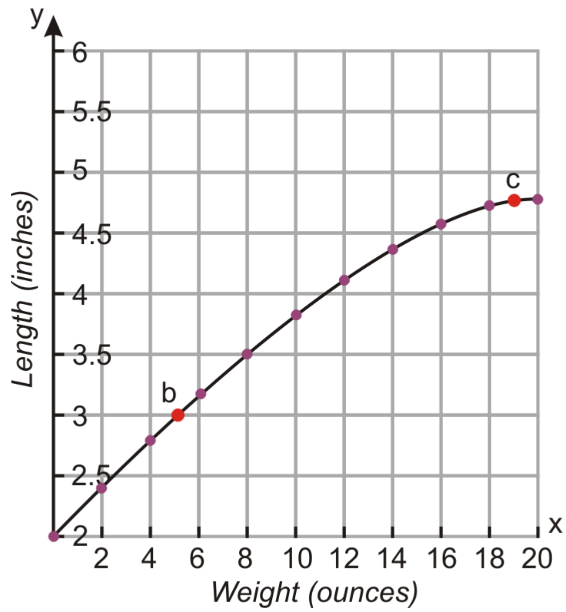
$$y = -.000145(5)^3 - .000221(5)^2 + .202(5) + 2.002 = 3 \ inches$$

c) To find the length of the spring when the weight is 19 ounces, we plug in $x = 19$.

$$y = -.000145(19)^3 - .000221(19)^2 + .202(19) + 2.002 = 4.77 \ inches$$

Step 6 Check

To check the validity of the solutions lets plot the answers to b) and c) on the scatter plot. We see that the answer to both b) and c) are close to the rest of the data.



We conclude that a cubic function represents the stretching of the spring more accurately than a linear function. However, for small weights the linear function is an equally good representation, and it is much easier to use in most cases. In fact, the linear approximation usually allows us to easily solve many problems that would be very difficult to solve by using the cubic function.

Example 2 Water flow

A thin cylinder is filled with water to a height of 50 centimeters. The cylinder has a hole at the bottom which is covered with a stopper. The stopper is released at time $t = 0$ seconds and allowed to empty. The following data shows the height of the water in the cylinder at different times.

Time(sec)	0	2	4	6	8	10	12	14	16	18	20	22	24
Height(cm)	50	42.5	35.7	29.5	23.8	18.8	14.3	10.5	7.2	4.6	2.5	1.1	0.2

a) Find the height (in centimeters) of water in the cylinder as a function of time in seconds.

b) Find the height of the water when $t = 5$ seconds.

c) Find the height of the water when $t = 13$ seconds.

Solution:

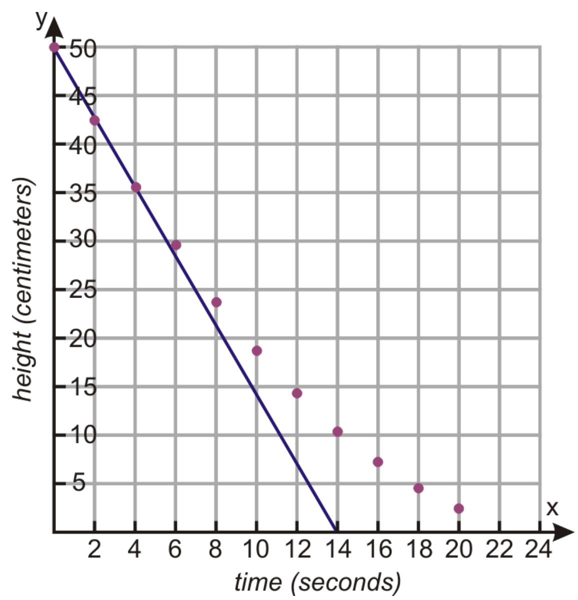
Step 1 Understand the problem

Define x = the time in seconds

y = height of the water in centimeters

Step 2 Devise a plan

Let's make a scatter plot of our data with the time on the horizontal axis and the height of water on the vertical axis.



Notice that most of the points seem to fit on a straight line when the water level is high. Assume that a function relating the height of the water to the time is linear.

Step 3 Solve

Find the equation of the line using points describing lighter weights:

(0, 50) and (4, 35.7).

The slope is $m = \frac{-14.3}{4} = -3.58$

Using $y = mx + b$

a) We obtain the function: $y = -3.58x + 50$

b) The height of the water when $t = 5$ seconds is

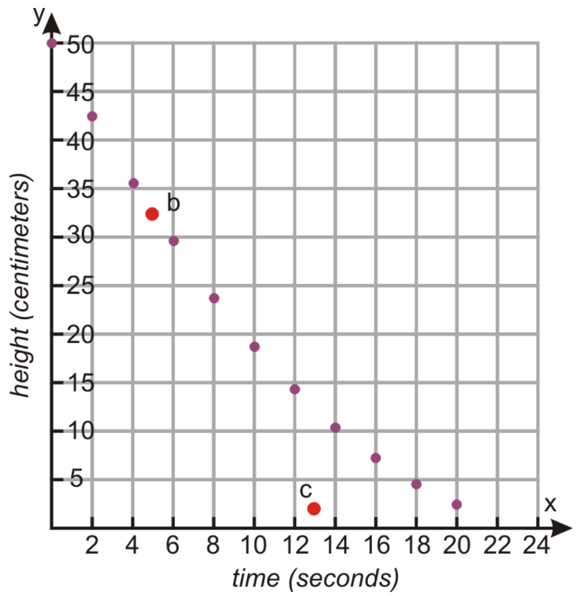
$$y = -3.58(5) + 50 = 32.1 \ centimeters$$

c) The height of the water when $t = 13$ seconds is

$$y = -3.58(13) + 50 = 3.46 \ centimeters$$

Step 4 Check

To check the validity of the solutions, let's plot the answers to b) and c) on the scatter plot. We see that the answer to b) is close to the rest of the data, but the answer to c) does not seem to follow the trend.



We can conclude that when the water level is high, the relationship between the height of the water and the time is a linear function. When the water level is low, we must change our assumption. There must be a non-linear relationship between the height and the time.

Step 5 Solve with new assumptions

Let's assume the relationship is quadratic and let's find the equation of the function by quadratic regression with a graphing calculator.

a) We obtain the function $y = .075x^2 - 3.87x + 50$

b) The height of the water when $t = 5$ seconds is

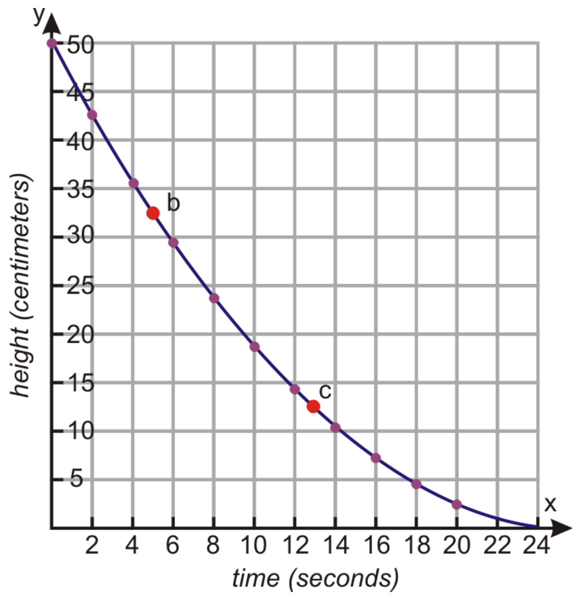
$$y = .075(5)^2 - 3.87(5) + 50 = 32.53 \ centimeters$$

c) The height of the water when $t = 13$ seconds is

$$y = .075(13)^2 - 3.87(13) + 50 = 12.37 \ centimeters$$

Step 6: Check

To check the validity of the solutions lets plot the answers to b) and c) on the scatterplot. We see that the answer to both b) and c) are close to the rest of the data.



We conclude that a quadratic function represents the situation more accurately than a linear function. However, for high water levels the linear function is an equally good representation.

Example 3 Projectile motion

A golf ball is hit down a straight fairway. The following table shows the height of the ball with respect to time. The ball is hit at an angle of 70 degrees with the horizontal with a speed of 40 meters/sec.

Time (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0
Height (meters)	0	17.2	31.5	42.9	51.6	57.7	61.2	62.3	61.0	57.2	51.0	42.6	31.9	19.0	4.1

a) Find the height of the ball as a function of time.

b) Find the height of the ball when $t = 2.4$ seconds.

c) Find the height of the ball when $t = 6.2$ seconds.

Solution

Step 1 Understand the problem

Define x = the time in seconds

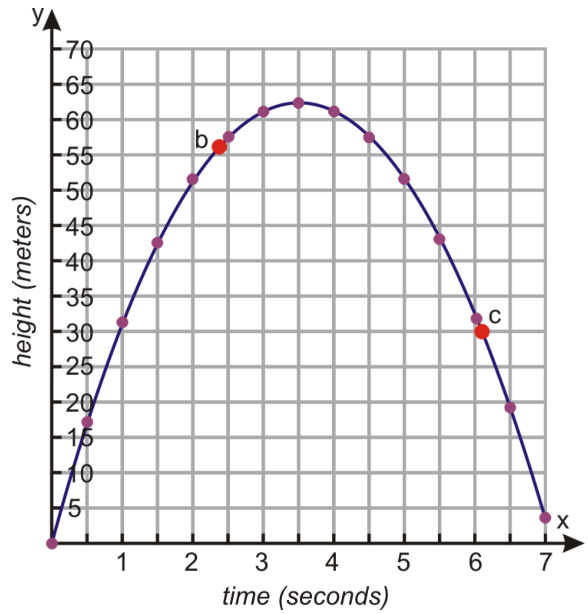
y = height of the ball in meters

Step 2 Devise a plan

Let's make a scatter plot of our data with the time on the horizontal axis and the height of the ball on the vertical axis. We know that a projectile follows a parabolic path, so we assume that the function relating height to time is quadratic.

Step 3 Solve

Let's find the equation of the function by quadratic regression with a graphing calculator.



a) We obtain the function $y = -4.92x^2 + 34.7x + 1.2$

b) The height of the ball when $t = 2.4$ seconds is:

$$y = -4.92(2.4)^2 + 34.7(2.4) + 1.2 = 56.1 \ meters$$

c) The height of the ball when $t = 6.2$ seconds is:

$$y = -4.92(6.2)^2 + 34.7(6.2) + 1.2 = 27.2 \ meters$$

Step 4 Check

To check the validity of the solutions lets plot the answers to b) and c) on the scatterplot. We see that the answer to both b) and c) follow the trend very closely. The quadratic function is a very good model for this problem

Example 4 Population growth

A scientist counts two thousand fish in a lake. The fish population increases at a rate of 1.5 fish per generation but the lake has space and food for only 2,000,000 fish. The following table gives the number of fish (in thousands) in each generation.

Generation	0	4	8	12	16	20	24	28
Number (thousands)	2	15	75	343	1139	1864	1990	1999

a) Find the number of fish as a function of generation.

b) Find the number of fish in generation 10.

c) Find the number of fish in generation 25.

Solution:

Step 1 Understand the problem

Define x = the generation number y = the number of fish in the lake

Step 2 Devise a plan

Let's make a scatterplot of our data with the generation number on the horizontal axis and the number of fish in the lake on the vertical axis. We know that a population can increase exponentially. So, we assume that we can use an exponential function to describe the relationship between the generation number and the number of fish.

Step 3 Solve

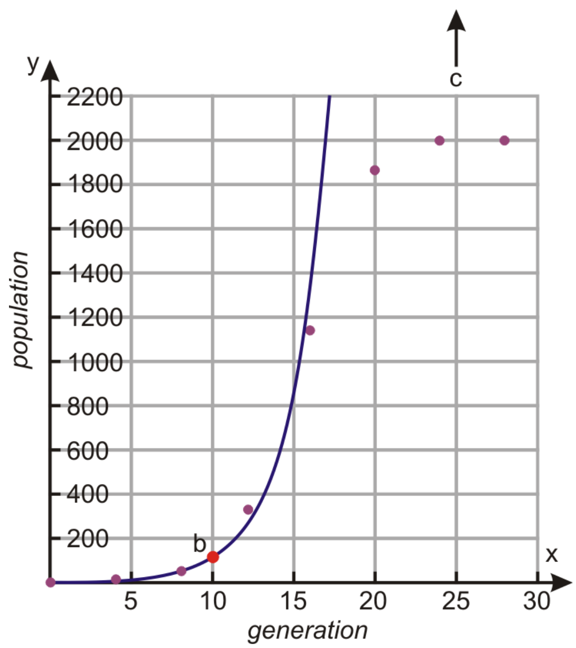
a) Since the population increases at a rate of 1.5 per generation, assume the function $y = 2(1.5)x$

b) The number of fish in generation 10 is: $y = 2(1.5)^{10} = 115$ thousand fish

c) The number of fish in generation 25 is: $y = 2(1.5)^{25} = 50502$ thousand fish

Step 4 Check

To check the validity of the solutions, let's plot the answers to b) and c) on the scatter plot. We see that the answer to b) fits the data well but the answer to c) does not seem to follow the trend very closely. The result is not even on our graph!



When the population of fish is high, the fish compete for space and resources so they do not increase as fast. We must change our assumptions.

Step 5 Solve with new assumptions

When we try different regressions with the graphing calculator, we find that logistic regression fits the data the best.

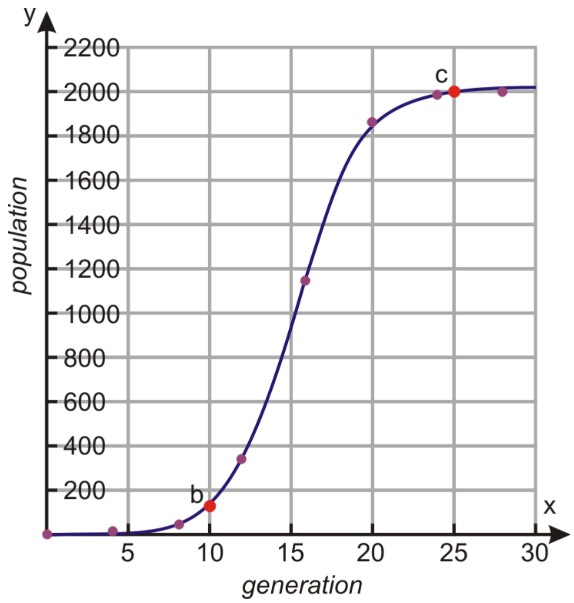
a) We obtain the function $y - \frac{2023.6}{1+1706.3(2.71)^{-484x}}$

b) The number of fish in generation 10 is $y = \frac{2023.6}{1+1706.3(2.71)^{-.484(10)}} = 139.6$ thousand fish

c) The number of fish in generation 25 is $y = \frac{2023.6}{1+1706.3(2.71)^{-.484(25)}} = 2005$ thousand fish

Step 6 Check

To check the validity of the solutions, let's plot the answers to b) and c) on the scatter plot. We see that the answer to both b) and c) are close to the rest of the data.



We conclude that a logistic function represents the situation more accurately than an exponential function. However, for small populations the exponential function is an equally good representation, and it is much easier to use in most cases.

Review Questions

1. In Example 1, evaluate the length of the spring for weight $= 3$ *ounces* by
   (a) Using the linear function
   (b) Using the cubic function
   (c) Figuring out which function is best to use in this situation.
2. In Example 1, evaluate the length of the spring for weight $= 15$ *ounces* by
   (a) Using the linear function
   (b) Using the cubic function
   (c) Figuring out which function is best to use in this situation.
3. In Example 2, evaluate the height of the water in the cylinder when $t = 4.2$ *seconds* by
   (a) Using the linear function
   (b) Using the quadratic function
   (c) Figuring out which function is best to use in this situation.
4. In Example 2, evaluate the height of the water in the cylinder when $t = 19$ *seconds* by
   (a) Using the linear function
   (b) Using the quadratic function
   (c) Figuring out which function is best to use in this situation.
5. In Example 3, evaluate the height of the ball when $t = 5.2$ *seconds*. Find when the ball is at its highest point.
6. In Example 4, evaluate the number of fish in generation 8 by
   (a) Using the exponential function
   (b) Using the logistic function
   (c) Figuring out which function is best to use in this situation.
7. In Example 4, evaluate the number of fish in generation 18 by
   (a) Using the exponential function

(b) Using the logistic function
(c) Figuring out which function is best to use in this situation.

Review Answers

1. (a) 2.6 inches
   (b) 2.6 inches
   (c) Both functions give the same result. The linear function is best because it is easier to use.
2. (a) 5 inches
   (b) 4.5 inches
   (c) The two functions give different answers. The cubic function is better because it gives a more accurate answer.
3. (a) 34.96 cm
   (b) 35.07 cm
   (c) The results from both functions are almost the same. The linear function is best because it is easier to use.
4. (a) -18.02 cm
   (b) 3.5 cm
   (c) The two function give different results. The quadratic function is better because it gives a more accurate answer.
5. (a) 48.6 meters
   (b) 3.7 seconds
6. (a) 51,000
   (b) 55,000
   (c) The results from both functions are almost the same. The linear function is best because it is easier to use.
7. (a) 2,956,000
   (b) 1,571,000
   (c) the two functions give different results; the logistic function is better because it gives a more accurate answer.

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See http://www.ck12.org/flexr/chapter/9620.

CHAPTER 4 Rational Equations and Functions

Chapter Outline

4.1 Inverse Variation Models

Learning Objectives

- Distinguish direct and inverse variation.
- Graph inverse variation equations.
- Write inverse variation equations.
- Solve real-world problems using inverse variation equations.

Introduction

Many variables in real-world problems are related to each other by variations. A **variation** is an equation that relates a variable to one or more variables by the operations of multiplication and division. There are three different kinds of variation problems: **direct variation, inverse variation and joint variation.**

Distinguish Direct and Inverse Variation

In **direct variation** relationships, the related variables will either increase together or decrease together at a steady rate. For instance, consider a person walking at three miles per hour. As time increases, the distance covered by the person walking also increases at the rate of three miles each hour. The distance and time are related to each other by a direct variation.

distance $=$ rate $\times$ time

Since the speed is a constant 3 miles per hour, we can write: $d = 3t$.

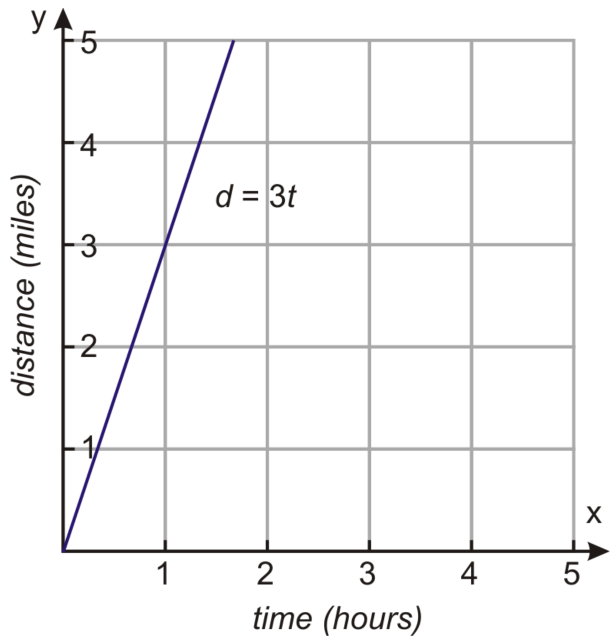
Direct Variation

The general equation for a direct variation is

$$y = kx.$$

k is called the **constant of proportionality**

You can see from the equation that a direct variation is a linear equation with a $y-$intercept of zero. The graph of a direct variation relationship is a straight line passing through the origin whose slope is k the constant of proportionality.



A second type of variation is **inverse variation**. When two quantities are related to each other inversely, as one quantitiy increases, the other one decreases and vice-versa.

For instance, if we look at the formula distance = speed × time again and solve for time, we obtain:

$$\text{time} = \frac{\text{distance}}{\text{rate}}$$

If we keep the distance constant, we see that as the speed of an object increases, then the time it takes to cover that distance decreases. Consider a car traveling a distance of 90 miles, then the formula relating time and speed is $t = \frac{90}{s}$.

Inverse Variation

The general equation for inverse variation is

$$y = \frac{k}{x}$$

where k is called the **constant of proportionality.**

In this chapter, we will investigate how the graph of these relationships behave.

Another type variation is a **joint variation.** In this type of relationship, one variable may vary as a product of two or more variables.

For example, the volume of a cylinder is given by:

$$V = \pi r^2 \cdot h$$

In this formula, the volume varies directly as the product of the square of the radius of the base and the height of the cylinder. The constant of proportionality here is the number π.

In many application problems, the relationship between the variables is a combination of variations. For instance Newtons Law of Gravitation states that the force of attraction between two spherical bodies varies jointly as the masses of the objects and inversely as the square of the distance between them

$$F = G\frac{m_1 m_2}{d^2}$$

In this example the constant of proportionality, G, is called the gravitational constant and its value is given by $G = 6.673 \times 10^{-11} N \cdot m^2/kg^2$.

Graph Inverse Variation Equations

We saw that the general equation for inverse variation is given by the formula $y = \left(\frac{k}{x}\right)$, where k is a constant of proportionality. We will now show how the graphs of such relationships behave. We start by making a table of values. In most applications, x and y are positive. So in our table, we will choose only positive values of x.

Example 1

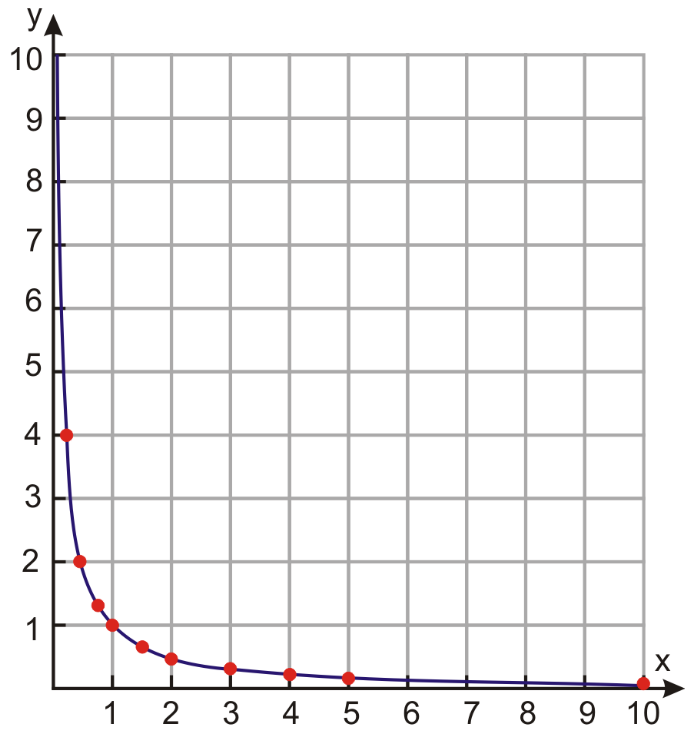
Graph an inverse variation relationship with the proportionality constant $k = 1$.

Solution

TABLE 4.1:

x	$y = \frac{1}{x}$
0	$y = \frac{1}{0} =$ undefined
$\frac{1}{4}$	$y = \frac{1}{\frac{1}{4}} = 4$
$\frac{1}{2}$	$y = \frac{1}{\frac{1}{2}} = 2$
$\frac{3}{4}$	$y = \frac{1}{\frac{3}{4}} = 1.33$
1	$y = \frac{1}{1} = 1$
$\frac{3}{2}$	$y = \frac{1}{\frac{3}{2}} = 0.67$
2	$y = \frac{1}{2} = 0.5$
3	$y = \frac{1}{3} = 0.33$
4	$y = \frac{1}{4} = 0.25$
5	$y = \frac{1}{5} = 0.2$
10	$y = \frac{1}{10} = 0.1$

Here is a graph showing these points connected with a smooth curve.



Both the table and the graph demonstrate the relationship between variables in an inverse variation. As one variable increases, the other variable decreases and vice-versa. Notice that when $x = 0$, the value of y is undefined. The graph shows that when the value of x is very small, the value of y is very big and it approaches infinity as x gets closer and closer to zero.

Similarly, as the value of x gets very large, the value of y gets smaller and smaller, but never reaches the value of zero. We will investigate this behavior in detail throughout this chapter

Write Inverse Variation Equations

As we saw an inverse variation fulfills the equation: $y = \left(\frac{k}{x}\right)$. In general, we need to know the value of y at a particular value of x in order to find the proportionality constant. After the proportionality constant is known, we can find the value of y for any given value of x.

Example 2

If y is inversely proportional to x and $y = 10$ *when* $x = 5$. *Find y when* $x = 2$.

Solution

Since y is inversely proportional to x,

then the general relationship tells us $\quad y = \frac{k}{x}$

Plug in the values $y = 10$ and $x = 5$. $\quad 10 = \frac{k}{5}$

Solve for k by multiplying both sides of the equation by 5. $\quad k = 50$

Now we put k back into the general equation.

The inverse relationship is given by $\quad y = \frac{50}{x}$

When $x = 2$ $\quad y = \frac{50}{2}$ or $y = 25$

Answer $y = 25$

Example 3

If p is inversely proportional to the square of q, and $p = 64$ *when* $q = 3$. *Find p when* $q = 5$.

Solution:

Since p is inversely proportional to q^2,	
then the general equation is	$p = \frac{k}{q^2}$
Plug in the values $p = 64$ and $q = 3$.	$64 = \frac{k}{3^2}$ or $64 = \frac{k}{9}$
Solve for k by multiplying both sides of the equation by 9.	$k = 576$
The inverse relationship is given by	$p = \frac{576}{q^2}$
When $q = 5$	$p = \frac{576}{25}$ or $p = 23.04$

Answer $p = 23.04$.

Solve Real-World Problems Using Inverse Variation Equations

Many formulas in physics are described by variations. In this section we will investigate some problems that are described by inverse variations.

Example 4

The frequency, f, *of sound varies inversely with wavelength,* λ. *A sound signal that has a wavelength of 34 meters has a frequency of 10 hertz. What frequency does a sound signal of 120 meters have?*

Solution

The inverse variation relationship is	$f = \frac{k}{\lambda}$
Plug in the values $\lambda = 34$ and $f = 10$.	$10 = \frac{k}{34}$
Multiply both sides by 34.	$k = 340$
Thus, the relationship is given by	$f = \frac{340}{\lambda}$
Plug in $\lambda = 120\ meters$.	$f = \frac{340}{120} \Rightarrow f = 2.83$

Answer $f = 2.83\ Hertz$

Example 5

Electrostatic force is the force of attraction or repulsion between two charges. The electrostatic force is given by the formula: $F = \left(\frac{Kq_1q_2}{d^2}\right)$ *where* q_1 *and* q_2 *are the charges of the charged particles,* d' *is the distance between the charges and k is proportionality constant. The charges do not change and are, thus, constants and can then be combined with the other constant k to form a new constant K. The equation is rewritten as* $F = \left(\frac{K}{d^2}\right)$. *If the electrostatic force is* $F = 740$ *Newtons when the distance between charges is* 5.3×10^{-11} *meters, what is F when* $d = 2.0 \times 10^{-10}$ *meters?*

Solution

The inverse variation relationship is	$f = \frac{k}{d^2}$
Plug in the values $F = 740$ and $d = 5.3 \times 10^{-11}$.	$740 = \frac{k}{(5.3 \times 10^{-11})^2}$
Multiply both sides by $(5.3 \times 10^{-11})^2$.	$K = 740\left(5.3 \times 10^{-11}\right)^2$
The electrostatic force is given by	$F = \frac{2.08 \times 10^{-18}}{d^2}$
When $d = 2.0 \times 10^{-10}$	$F = \frac{2.08 \times 10^{-18}}{(2.0 \times 10^{-10})^2}$
Enter $\frac{2.08 * 10^{(-18)}}{\left(2.0 * 10^{(-10)}\right)^2}$ into a calculator.	$F = 52$

Answer $F = 52\ Newtons$

Note: In the last example, you can also compute $F = \frac{2.08 \times 10^{-18}}{(2.0 \times 10^{-10})^2}$ by hand.

$$\begin{aligned} F &= \frac{2.08 \times 10^{-18}}{(2.0 \times 10^{-10})^2} \\ &= \frac{2.08 \times 10^{-18}}{4.0 \times 10^{-20}} \\ &= \frac{2.08 \times 10^{20}}{4.0 \times 10^{18}} \\ &= \frac{2.08}{4.0}\left(10^2\right) \\ &= 0.52(100) \\ &= 52 \end{aligned}$$

This illustrates the usefulness of scientific notation.

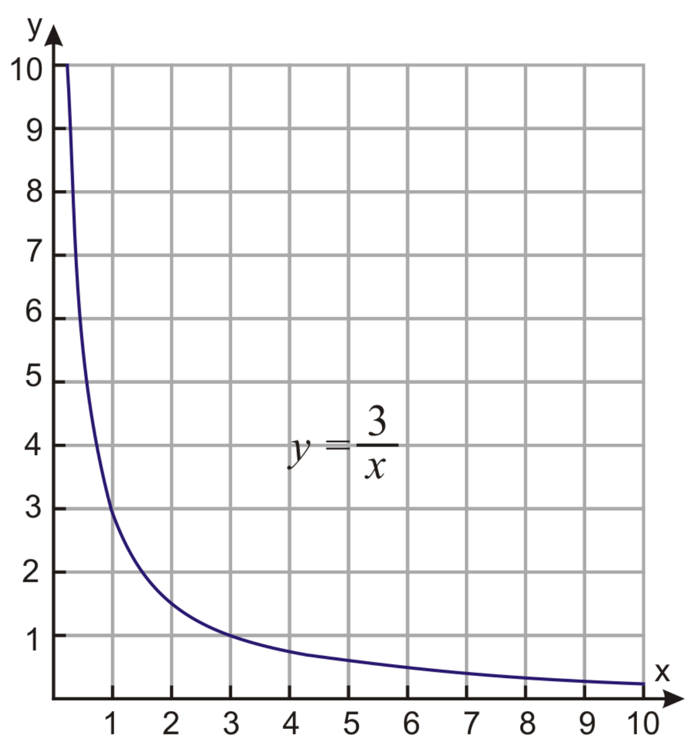
Review Questions

Graph the following inverse variation relationships.

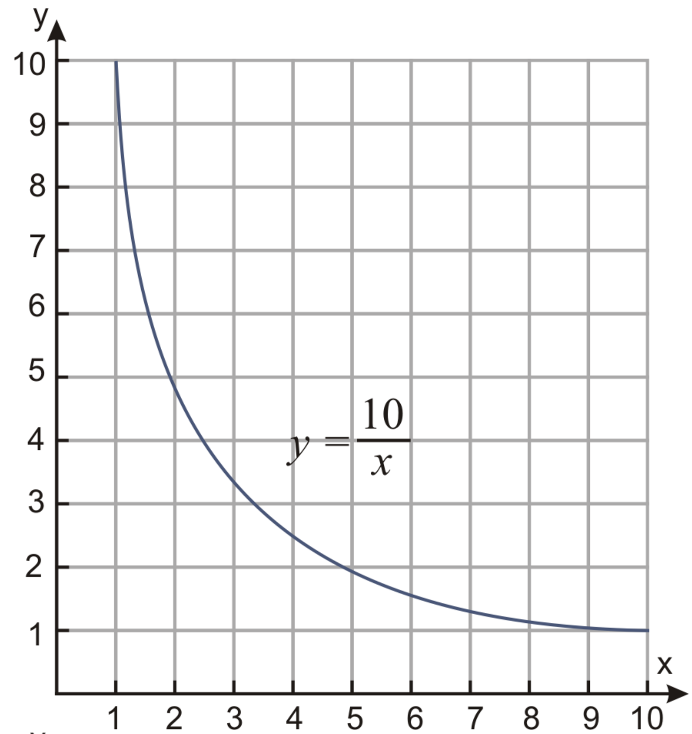
1. $y = \frac{3}{x}$
2. $y = \frac{10}{x}$
3. $y = \frac{1}{4x}$
4. $y = \frac{5}{6x}$
5. If z is inversely proportional to w and $z = 81$ when $w = 9$, find w when $z = 24$.
6. If y is inversely proportional to x and $y = 2$ when $x = 8$, find y when $x = 12$.
7. If a is inversely proportional to the square root of b, and $a = 32$ when $b = 9$, find b when $a = 6$.
8. If w is inversely proportional to the square of u and $w = 4$ when $u = 2$, find w when $u = 8$.
9. If x is proportional to y and inversely proportional to z, and $x = 2$, when $y = 10$ and $z = 25$. Find x when $y = 8$ and $z = 35$.
10. If a varies directly with b and inversely with the square of c and $a = 10$ when $b = 5$ and $c = 2$. Find the value of a when $b = 3$ and $c = 6$.

11. The intensity of light is inversely proportional to the square of the distance between the light source and the object being illuminated. A light meter that is 10 meters from a light source registers 35 lux. What intensity would it register 25 meters from the light source?
12. Ohms Law states that current flowing in a wire is inversely proportional to the resistance of the wire. If the current is 2.5 Amperes when the resistance is 20 ohms, find the resistance when the current is 5 Amperes.
13. The volume of a gas varies directly to its temperature and inversely to its pressure. At 273 degrees Kelvin and pressure of 2 atmospheres, the volume of the gas is 24 Liters. Find the volume of the gas when the temperature is 220 kelvin and the pressure is 1.2 atmospheres.
14. The volume of a square pyramid varies jointly as the height and the square of the length of the base. A cone whose height is 4 inches and whose base has a side length of 3 inches has a volume of 12 in^3. Find the volume of a square pyramid that has a height of 9 inches and whose base has a side length of 5 inches.

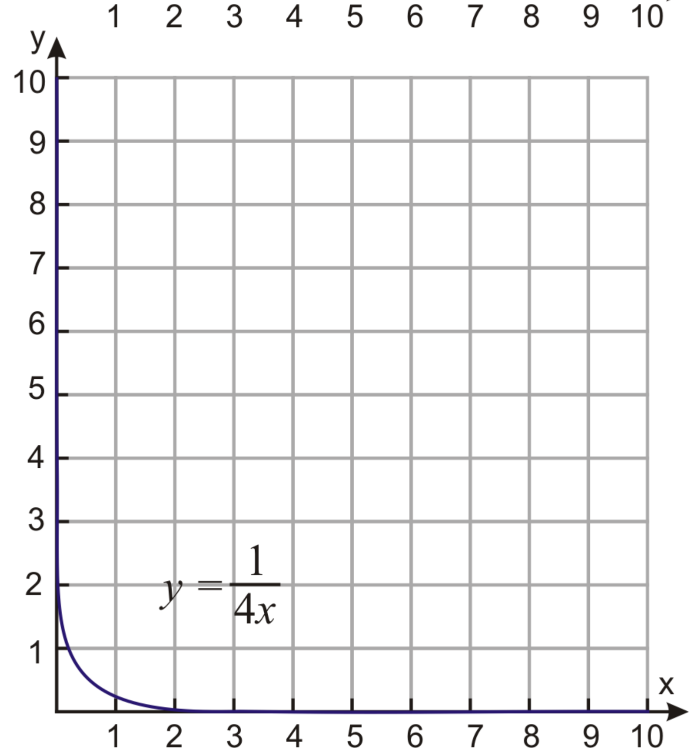
Review Answers

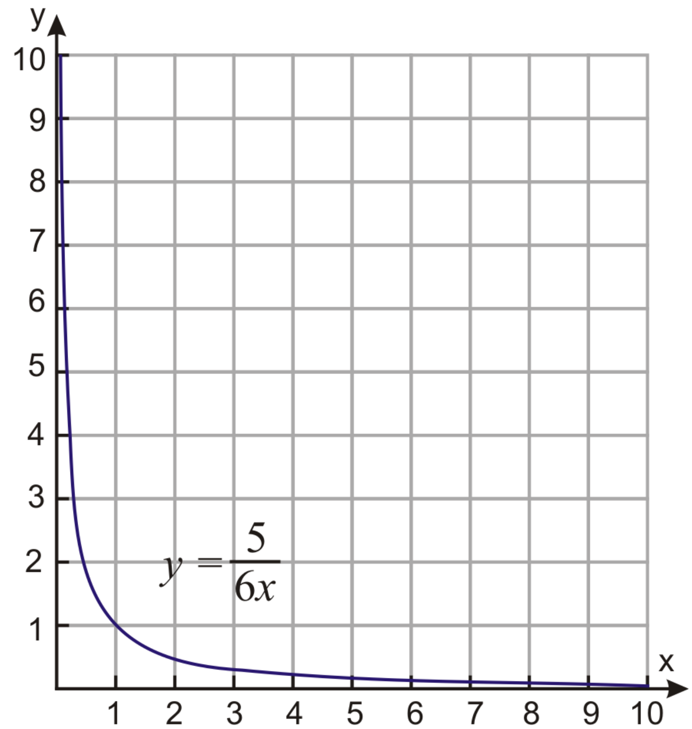
1. 

2.


y
10
9
8
7
6
5
4
3
2
1
$y = \frac{10}{x}$
x
1 2 3 4 5 6 7 8 9 10

3.


y
10
9
8
7
6
5
4
3
2
1
$y = \frac{1}{4x}$
x
1 2 3 4 5 6 7 8 9 10

4. 

5. $W = \frac{243}{8}$
6. $y = \frac{4}{3}$
7. $b = 256$
8. $w = \frac{1}{4}$
9. $x = \frac{8}{7}$
10. $a = \frac{2}{3}$
11. $I = 5.6\ lux$
12. $R = 10\ ohms$
13. $V = 32.2\ L$
14. $V = 75\ in^3$

4.2 Graphs of Rational Functions

Learning Objectives

- Compare graphs of inverse variation equations.
- Graph rational functions.
- Solve real-world problems using rational functions.

Introduction

In this section, you will learn how to graph rational functions. Graphs of rational functions are very distinctive. These functions are characterized by the fact that the function gets closer and closer to certain values but never reaches those values. In addition, because rational functions may contain values of x where the function does not exit, the function can take values very close to the excluded values but never cross through these values. This behavior is called asymptotic behavior and we will see that rational functions can have **horizontal asymptotes, vertical asymptotes** or **oblique (or slant) asymptotes**.

Compare Graphs of Inverse Variation Equations

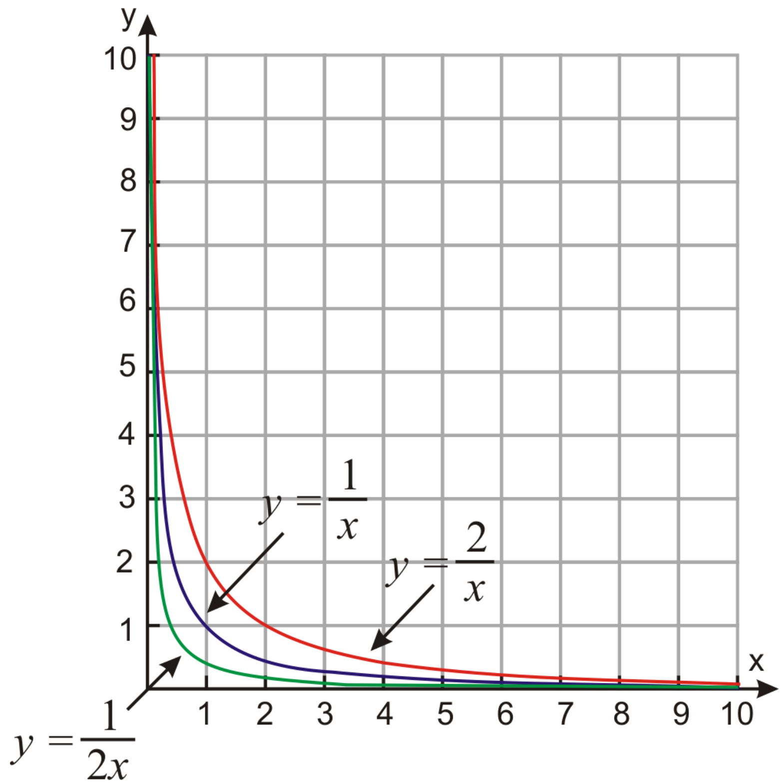
Inverse variation problems are the simplest example of rational functions. We saw that an inverse variation has the general equation: $y = \frac{k}{x}$ or $f(x) = \frac{k}{x}$. In most real-world problems, the x and y values take only positive values. Below, we will show graphs of three inverse variation functions.

Example 1

On the same coordinate grid, graph an inverse variation relationships with the proportionality constants $k = 1, k = 2$, *and* $k = \left(\frac{1}{2}\right)$.

Solution

We will not show the table of values for this problem, but rather we can show the graphs of the three functions on the same coordinate axes. We notice that for larger constants of proportionality, the curve decreases at a slower rate than for smaller constants of proportionality. This makes sense because, basically the value of y is related directly to the proportionality constants so we should expect larger values of y for larger values of k.



Graph Rational Functions

We will now extend the domain and range of rational equations to include negative values of x and y. We will first plot a few rational functions by using a table of values, and then we will talk about distinguishing characteristics of rational functions that will help us make better graphs.

Recall that one of the basic rules of arithmetic is that you cannot divide by 0.

$$\frac{0}{5} = 0$$

while

$$\frac{5}{0} = \text{Undefined.}$$

As we graph rational functions, we need to always pay attention to values of x that will cause us to divide by 0.

Example 2

Graph the function $f(x) = \frac{1}{x}$.

Solution

Before we make a table of values, we should notice that the function is not defined for $x = 0$. This means that the graph of the function will not have a value at that point. Since the value of $x = 0$ is special, we should make sure to pick enough values close to $x = 0$ in order to get a good idea how the graph behaves. Lets make two tables: one for $x-$values smaller than zero and one for $x-$values larger than zero. For the table of values it may be helpful to replace f(x) with y.

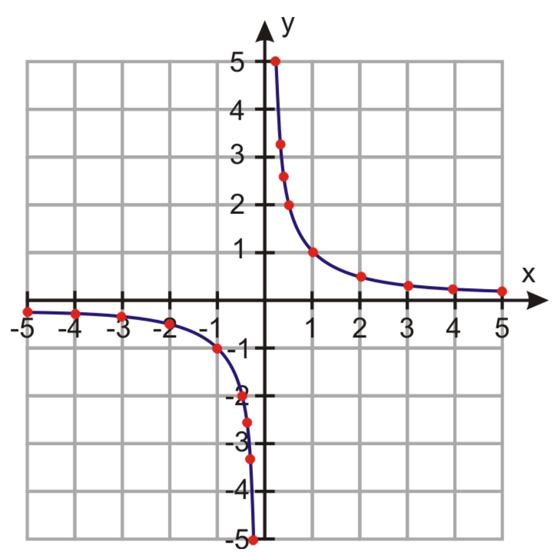
TABLE 4.2:

x	$y = \frac{1}{x}$
-5	$y = \frac{1}{-5} = -0.2$
-4	$y = \frac{1}{-4} = -0.25$
-3	$y = \frac{1}{-3} = -0.33$
-2	$y = \frac{1}{-2} = -0.5$
-1	$y = \frac{1}{-1} = -1$
-0.5	$y = \frac{1}{-0.5} = -2$
-0.4	$y = \frac{1}{-0.4} = -2.5$
-0.3	$y = \frac{1}{-0.3} = -3.3$
-0.2	$y = \frac{1}{-0.2} = -5$
-0.1	$y = \frac{1}{-0.1} = -10$

TABLE 4.3:

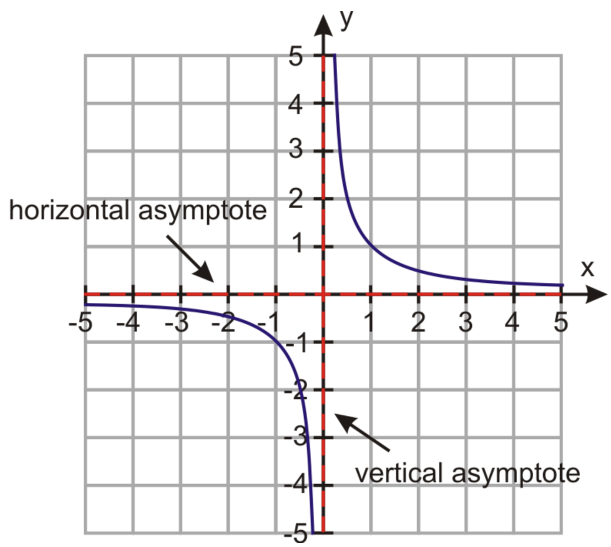
x	$y = \frac{1}{x}$
0.1	$y = \frac{1}{0.1} = 10$
0.2	$y = \frac{1}{0.2} = 5$
0.3	$y = \frac{1}{0.3} = 3.3$
0.4	$y = \frac{1}{0.4} = 2.5$
0.5	$y = \frac{1}{0.5} = 2$
1	$y = \frac{1}{1} = 1$
2	$y = \frac{1}{2} = 0.5$
3	$y = \frac{1}{3} = 0.33$
4	$y = \frac{1}{4} = 0.25$
5	$y = \frac{1}{5} = 0.2$

We can see in the table that as we pick positive values of x closer and closer to zero, y becomes increasing large. As we pick negative values of x closer and close to zero, y becomes increasingly small (or more and more negative).



Notice on the graph that for values of x near 0, the points on the graph get closer and closer to the vertical line $x = 0$. The line $x = 0$ is called a **vertical asymptote** of the function $f(x) = \left(\frac{1}{x}\right)$.

We also notice that as x gets larger in the positive direction or in the negative direction, the value of y gets closer and closer to, but it will never actually equal zero. Why? Since $f(x) = \left(\frac{1}{x}\right)$, there are **no** values of x that will make the fraction zero. For a fraction to equal zero, the numerator must equal zero. The horizontal line $y = 0$ is called a **horizontal asymptote** of the function $f(x) = \left(\frac{1}{x}\right)$.



Asymptotes are usually denoted as dashed lines on a graph. They are not part of the function. A vertical asymptote shows that the function cannot take the value of x represented by the asymptote. A horizontal asymptote shows the value of y that the function approaches for large absolute values of x.

Here we show the graph of our function with the vertical and horizontal asymptotes drawn on the graph.

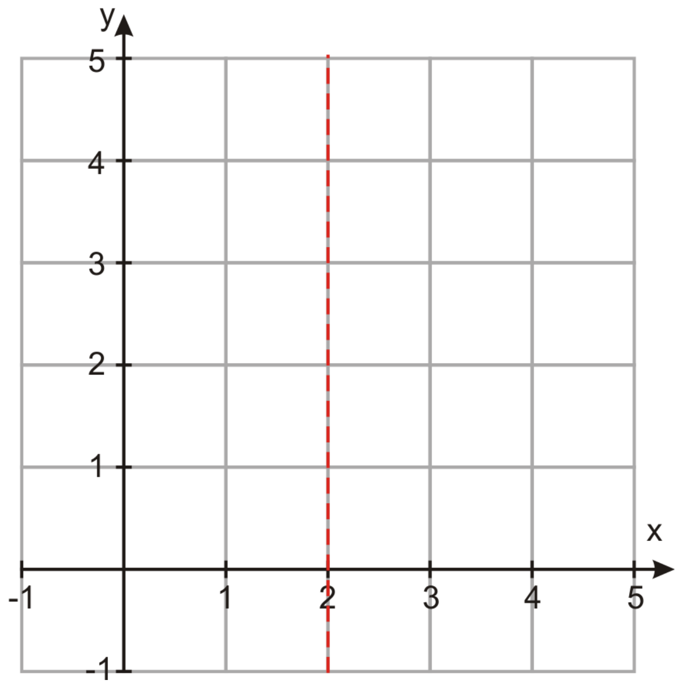
Next we will show the graph of a rational function that has a vertical asymptote at a non-zero value of x.

Example 3

Graph the function $f(x) = \frac{1}{(x-2)^2}$.

Solution

Before we make a table of values we can see that the function is not defined for $x = 2$ because that will cause division by 0. This tells us that there should be a vertical asymptote at $x = 2$. We start graphing the function by drawing the vertical asymptote.

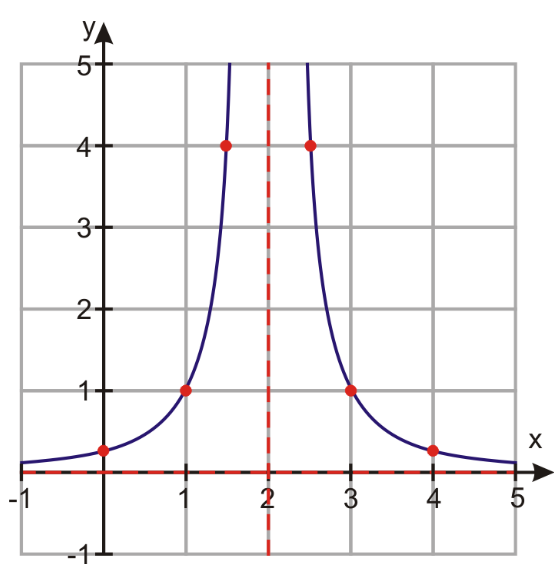


Now lets make a table of values.

TABLE 4.4:

x	$y=\frac{1}{(x-2)^2}$
0	$y=\frac{1}{(0-2)^2}=\frac{1}{4}$
1	$y=\frac{1}{(1-2)^2}=1$
1.5	$y=\frac{1}{(1.5-2)^2}=4$
2	undefined
2.5	$y=\frac{1}{(2.5-2)^2}=4$
3	$y=\frac{1}{(3-2)^2}=1$
4	$y=\frac{1}{(4-2)^2}=\frac{1}{4}$

Here is the resulting graph



Notice that we did not pick as many values for our table this time. This is because we should have a good idea what happens near the vertical asymptote. We also know that for large values of x, both positive and negative, the value of y could approach a constant value.

In this case, that constant value is $y = 0$. This is the horizontal asymptote.

A rational function does not need to have a vertical or horizontal asymptote. The next example shows a rational function with no vertical asymptotes.

Example 4

Graph the function $f(x) = \frac{x^2}{x^2+1}$.

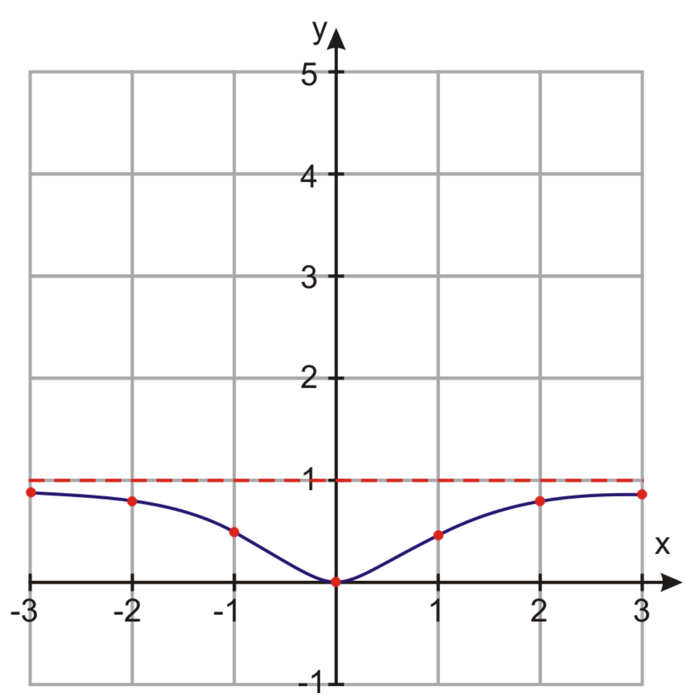
Solution

We can see that this function will have no vertical asymptotes because the denominator of the function will never be zero. Lets make a table of values to see if the value of y approaches a particular value for large values of x, both positive and negative.

TABLE 4.5:

x	$y = \frac{x^2}{x^2+1}$
-3	$y = \frac{(-3)^2}{(-3)^2+1} = \frac{9}{10} = 0.9$
-2	$y = \frac{(-2)^2}{(-2)^2+1} = \frac{4}{5} = 0.8$
-1	$y = \frac{(-1)^2}{(-1)^2+1} = \frac{1}{2} = 0.5$
0	$y = \frac{(0)^2}{(0)^2+1} = \frac{0}{1} = 0$
1	$y = \frac{(1)^2}{(1)^2+1} = \frac{1}{2} = 0.5$
2	$y = \frac{(2)^2}{(2)^2+1} = \frac{4}{5} = 0.8$
2	$y = \frac{(3)^2}{(3)^2+1} = \frac{9}{10} = 0.9$

Below is the graph of this function.



The function has no vertical asymptote. However, we can see that as the values of $|x|$ get larger the value of y get closer and closer to 1, so the function has a horizontal asymptote at $y = 1$.

More on Horizontal Asymptotes

We said that a horizontal asymptote is the value of y that the function approaches for large values of $|x|$. When we plug in large values of x in our function, higher powers of x get larger more quickly than lower powers of x. For example,

$$f(x) = \frac{2x^2 + x - 1}{3x^2 - 4x + 3}$$

If we plug in a large value of x, say $x = 100$, we obtain:

$$y = \frac{2(100)^2 + (100) - 1}{3(100)^2 - 4(100) + 3} = \frac{20000 + 100 - 1}{30000 - 400 + 2}$$

We can see that the first terms in the numerator and denominator are much bigger that the other terms in each expression. One way to find the horizontal asymptote of a rational function is to ignore all terms in the numerator and denominator except for the highest powers.

In this example the horizontal asymptote is $f(x) = \frac{2x^2}{3x^2}$ which simplifies to $y = \frac{2}{3}$.

In the function above, the highest power of x was the same in the numerator as in the denominator. Now consider a function where the power in the numerator is less than the power in the denominator.

$$f(x) = \frac{x}{x^2 + 3}$$

As before, we ignore all but the terms except the highest power of x in the numerator and the denominator.

Horizontal asymptote $y = \frac{x}{x^2}$ which simplifies to $y = \frac{1}{x}$

For large values of x, the value of y gets closer and closer to zero. Therefore the horizontal asymptote in this case is $y = 0$.

To Summarize

- Find vertical asymptotes by setting the denominator equal to zero and solving for x.
- For horizontal asymptotes, we must consider several cases for finding horizontal asymptotes.
  - If the highest power of x in the numerator is less than the highest power of x in the denominator, then the horizontal asymptote is at $y = 0$.
  - If the highest power of x in the numerator is the same as the highest power of x in the denominator, then the horizontal asymptote is at $y = \frac{\text{coefficient of highest power of } x}{\text{coefficient of highest power of } x}$
  - If the highest power of x in the numerator is greater than the highest power of x in the denominator, then we dont have a horizontal asymptote, we could have what is called an oblique (slant) asymptote or no asymptote at all.

Example 5

Find the vertical and horizontal asymptotes for the following functions.

a) $f(x) = \frac{1}{x-1}$

b) $f(x) = \frac{3x}{4x+2}$

c) $f(x) = \frac{x^2-2}{2x^2+3}$

d) $f(x) = \frac{x^3}{x^2-3x+2}$

Solution

a) **Vertical asymptotes**

Set the denominator equal to zero. $x-1=0 \Rightarrow x=1$ is the vertical asymptote.

Horizontal asymptote

Keep only highest powers of x. $y = \frac{1}{x} \Rightarrow y = 0$ is the horizontal asymptote.

b) **vertical asymptotes**

Set the denominator equal to zero. $4x+2=0 \Rightarrow x = -\frac{1}{2}$ is the vertical asymptote.

Horizontal asymptote

Keep only highest powers of x. $y = \frac{3x}{4x} \Rightarrow y = \frac{3}{4}$ is the horizontal asymptote.

c) **Vertical asymptotes**

Set the denominator equal to zero. $2x^2+3=0 \Rightarrow 2x^2=-3 \Rightarrow x^2 = -\frac{3}{2}$ Since there are no solutions to this equation therre is no vertical asymptote.

Horizontal asymptote

Keep only highest powers of x. $y = \frac{x^2}{2x^2} \Rightarrow y = \frac{1}{2}$ is the horizontal asymptote.

d) **Vertical asymptotes**

Set the denominator equal to zero. $x^2-3x+2=0$

Factor. $(x-2)(x-1)=0$

Solve. $x=2$ and $x=1$ vertical asymptotes

Horizontal asymptote. There is no horizontal asymptote because power of numerator is larger than the power of the denominator

Notice the function in part d of Example 5 had more than one vertical asymptote. Here is an example of another function with two vertical asymptotes.

Example 6

Graph the function $f(x) = \frac{-x^2}{x^2-4}$.

Solution

We start by finding where the function is undefined.

Lets set the denominator equal to zero. $x^2-4=0$

Factor. $(x-2)(x-2)=0$

Solve. $x=2, x=-2$

We find that the function is undefined for $x=2$ and $x=-2$, so we know that there are vertical asymptotes at these values of x.

We can also find the horizontal asymptote by the method we outlined above.

Horizontal asymptote is at $y = \frac{-x^2}{x^2}$ or $y=-1$.

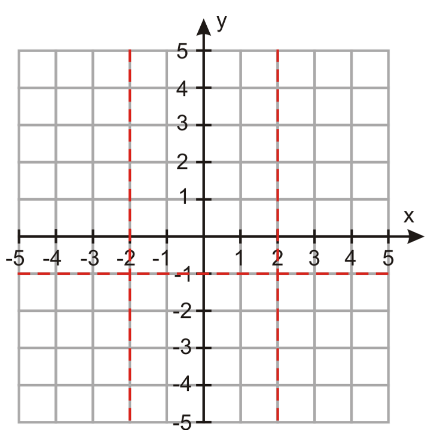
Start plotting the function by drawing the vertical and horizontal asymptotes on the graph.

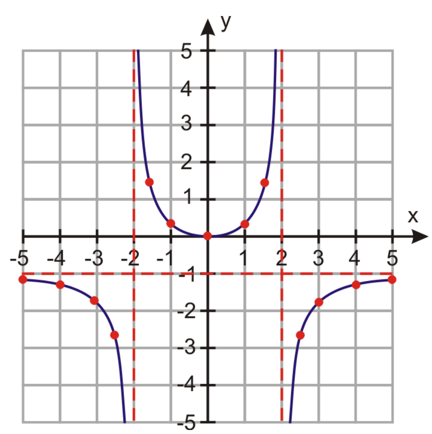
Now, lets make a table of values. Because our function has a lot of detail we must make sure that we pick enough values for our table to determine the behavior of the function accurately. We must make sure especially that we pick values close to the vertical asymptotes.

TABLE 4.6:

x	$y = \frac{-x^2}{x^2-4}$
-5	$y = \frac{-(-5)^2}{(-5)^2-4} = \frac{-25}{21} = -1.19$
-4	$y = \frac{-(-4)^2}{(-4)^2-4} = \frac{-16}{12} = -1.33$
-3	$y = \frac{-(-3)^2}{(-3)^2-4} = \frac{-9}{5} = -1.8$
-2.5	$y = \frac{-(-2.5)^2}{(-2.5)^2-4} = \frac{-6.25}{2.25} = -2.8$
-1.5	$y = \frac{-(-1.5)^2}{(-1.5)^2-4} = \frac{-2.25}{-1.75} = 1.3$
-1	$y = \frac{-(-1)^2}{(-1)^2-4} = \frac{-1}{-3} = 0.33$
-0	$y = \frac{(-0)^2}{(-0)^2-4} = \frac{0}{-4} = 0$
1	$y = \frac{-1^2}{(1)^2-4} = \frac{-1}{-3} = 0.33$
1.5	$y = \frac{-1.5^2}{(1.5)^2-4} = \frac{-2.25}{-1.75} = 1.3$
2.5	$y = \frac{-2.5^2}{(2.5)^2-4} = \frac{-6.25}{2.25} = -2.8$
3	$y = \frac{-3^2}{(-3)^2-4} = \frac{-9}{5} = -1.8$
4	$y = \frac{-4^2}{(-4)^2-4} = \frac{-16}{12} = -1.33$
5	$y = \frac{-5^2}{(-5)^2-4} = \frac{-25}{21} = -1.19$

Here is the resulting graph.

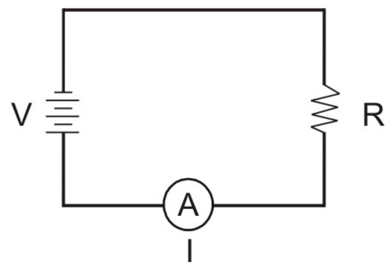




Solve Real-World Problems Using Rational Functions

Electrical Circuits

Electrical circuits are commonplace is everyday life. For instance, they are present in all electrical appliances in your home. The figure below shows an example of a simple electrical circuit. It consists of a battery which provides a voltage (V, measured in Volts, V), a resistor (R, measured in ohms, Ω) which resists the flow of electricity, and an ammeter that measures the current (I, measured in amperes, A) in the circuit.



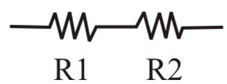
Ohms Law gives a relationship between current, voltage and resistance. It states that

$$I = \frac{V}{R}$$

Your light bulb, toaster and hairdryer are all basically simple resistors. In addition, resistors are used in an electrical circuit to control the amount of current flowing through a circuit and to regulate voltage levels. One important reason to do this is to prevent sensitive electrical components from burning out due to too much current or too high a voltage level. Resistors can be arranges in series or in parallel.

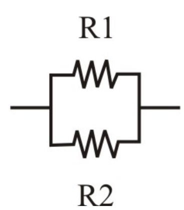
For resistors placed in a series, the total resistance is just the sum of the resistances of the individual resistors.

$R_{tot} = R_1 + R_2$



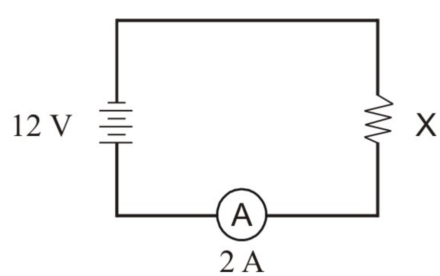
For resistors placed in parallel, the reciprocal of the total resistance is the sum of the reciprocals of the resistance of the individual resistors.

$\frac{1}{R_c} = \frac{1}{R_1} + \frac{1}{R_2}$



Example 7

Find the quantity labeled x in the following circuit.



Solution

We use the formula that relates voltage, current and resistance $I = \frac{V}{R}$

Plug in the known values $I = 2A, V = 12V$: $2 = \frac{12}{R}$

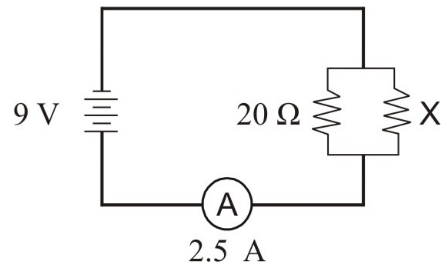
Multiply both sides by R. $2R = 12$

Divide both sides by 2. $R = 6\ \Omega$

Answer $6\ \Omega$

Example 8

Find the quantity labeled x in the following circuit.



Solution

Ohms Law also says $I_{total} = \frac{V_{total}}{R_{total}}$

Plug in the values we know, $I = 2.5\ A$ and $E = 9V$.

$2.5 = \frac{9}{R_{total}}$

Multiply both sides by R_{total}. $2.5R_{total} = 9$

Divide both sides by 2.5. $R_{tot} = 3.6\ \Omega$

Since the resistors are placed in parallel, the total resistance is given by

$$\frac{1}{R_{total}} = \frac{1}{x} + \frac{1}{20}$$
$$\Rightarrow \frac{1}{3.6} = \frac{1}{x} + \frac{1}{20}$$

Multiply all terms by $72X$. $\frac{1}{3.6}(72x) = \frac{1}{x}(72x) + \frac{1}{20}(72x)$

Cancel common factors. $20x = 72 + 3.6X$

Solve. $16.4x = 72$

Divide both sides by $16.4. x = 4.39\ \Omega$

Answer $x = 4.39\ \Omega$

Review Questions

Find all the vertical and horizontal asymptotes of the following rational functions.

1. $f(x) = \frac{4}{x+2}$
2. $f(x) = \frac{5x-1}{2x-6}$
3. $f(x) = \frac{10}{x}$
4. $f(x) = \frac{4x^2}{4x^2+1}$
5. $f(x) = \frac{2x}{x^2-9}$

6. $f(x) = \frac{3x^2}{x^2-4}$
7. $f(x) = \frac{1}{x^2+4x+3}$
8. $f(x) = \frac{2x+5}{x^2-2x-8}$

Graph the following rational functions. Draw dashed vertical and horizontal lines on the graph to denote asymptotes.

9. $f(x) = \frac{y}{2-x^3}$
10. $f(x) = \frac{3}{x^2}$
11. $f(x) = \frac{x}{x-1}$
12. $f(x) = \frac{2x}{x+1}$
13. $f(x) = \frac{-1}{x^2+2}$
14. $f(x) = \frac{x}{x^2+9}$
15. $f(x) = \frac{x^2}{x^2+1}$
16. $f(x) = \frac{1}{x^2-1}$
17. $f(x) = \frac{2x}{x^2-9}$
18. $f(x) = \frac{x^2}{x^2-16}$
19. $f(x) = \frac{3}{x^2-4x+4}$
20. $f(x) = \frac{x}{x^2-x-6}$

Find the quantity labeled x in the following circuit.

21.

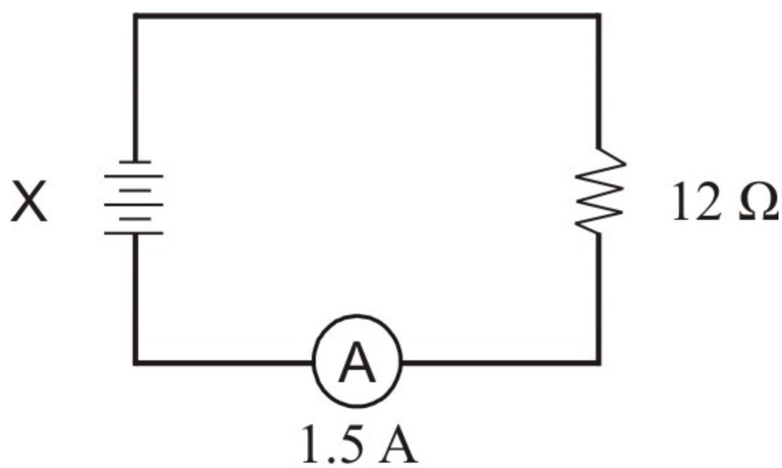


FIGURE 4.1

22.
23.
24.

Review Answers

1. vertical $x = -2$; horizontal $y = 0$

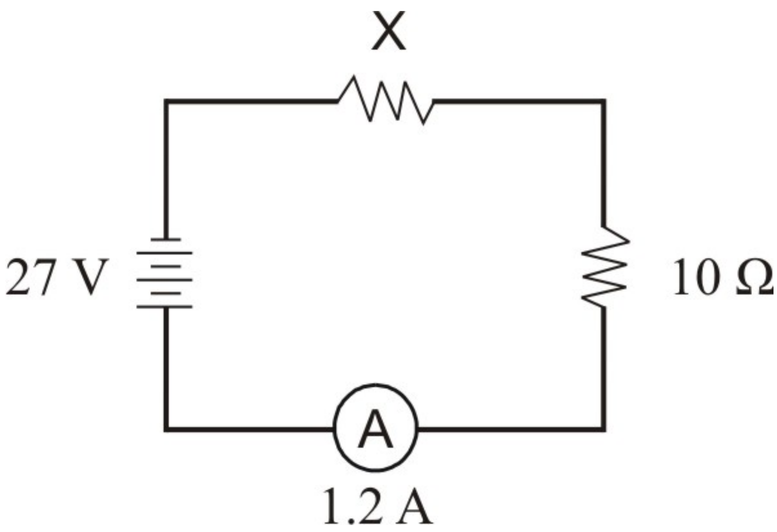


FIGURE 4.2

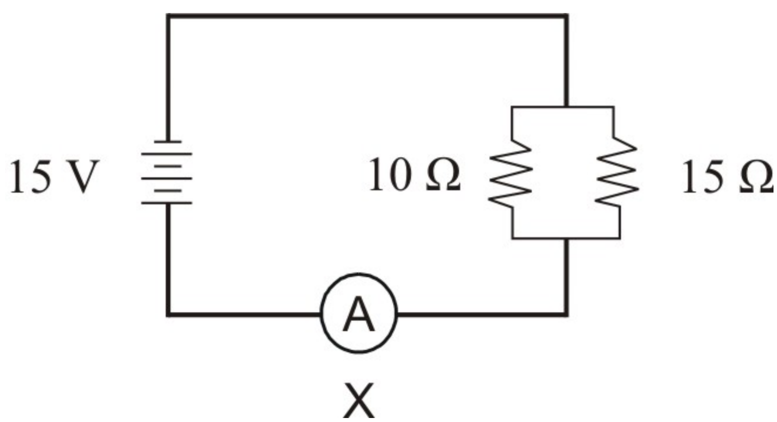


FIGURE 4.3

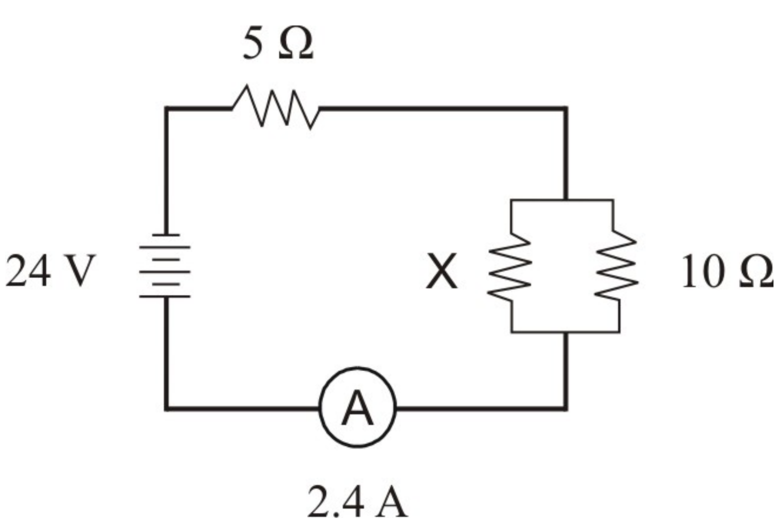


FIGURE 4.4

2. vertical $x = 3$; horizontal $y = \frac{5}{2}$
3. vertical $x = 0$; horizontal $y = 0$
4. no vertical; horizontal $y = 1$
5. vertical $x = 3, x = -3$; horizontal $y = 0$
6. vertical $x = 2, x = -2$; horizontal $y = 3$
7. vertical $x = -1, x = -3$; horizontal $y = 0$
8. vertical $x = 4, x = -2$; horizontal $y = 0$
9.

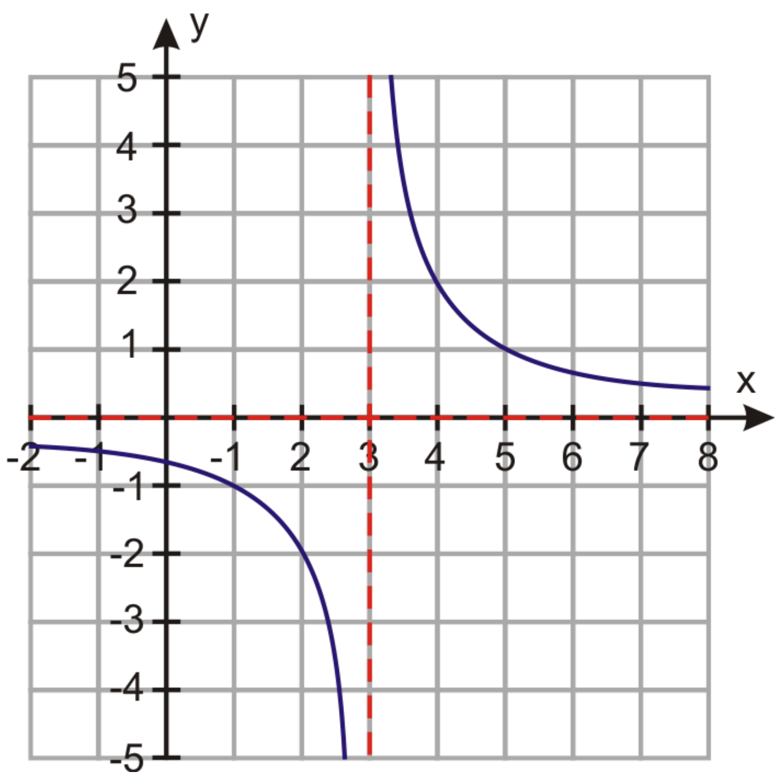


FIGURE 4.5

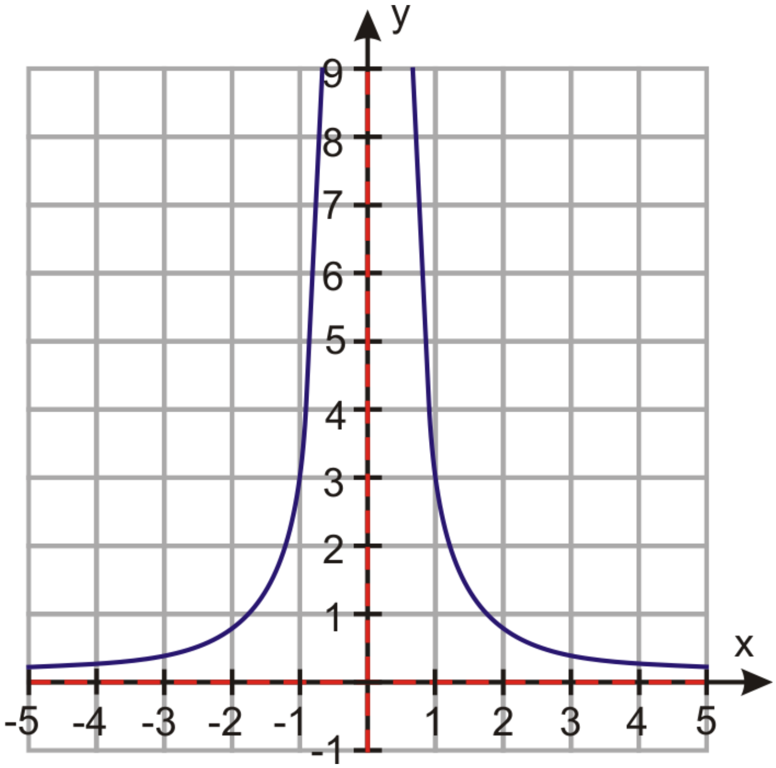
10. 

FIGURE 4.6

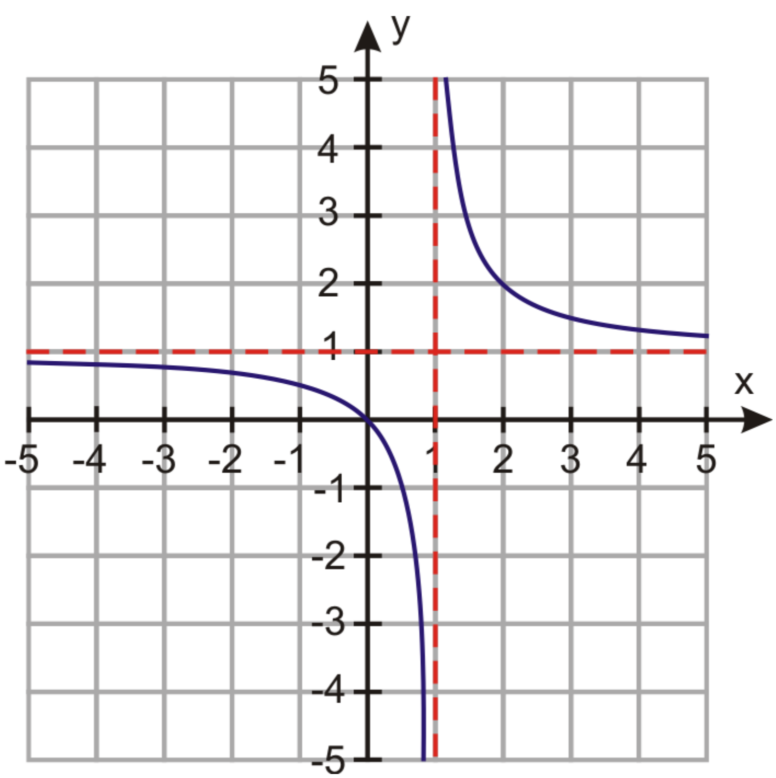
11. 

FIGURE 4.7

12.

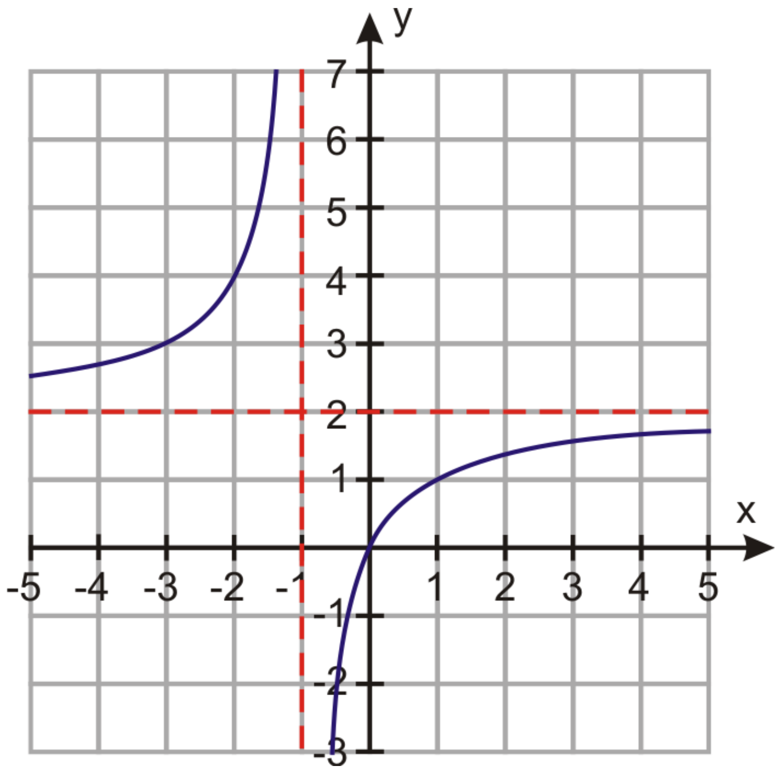


FIGURE 4.8

13.

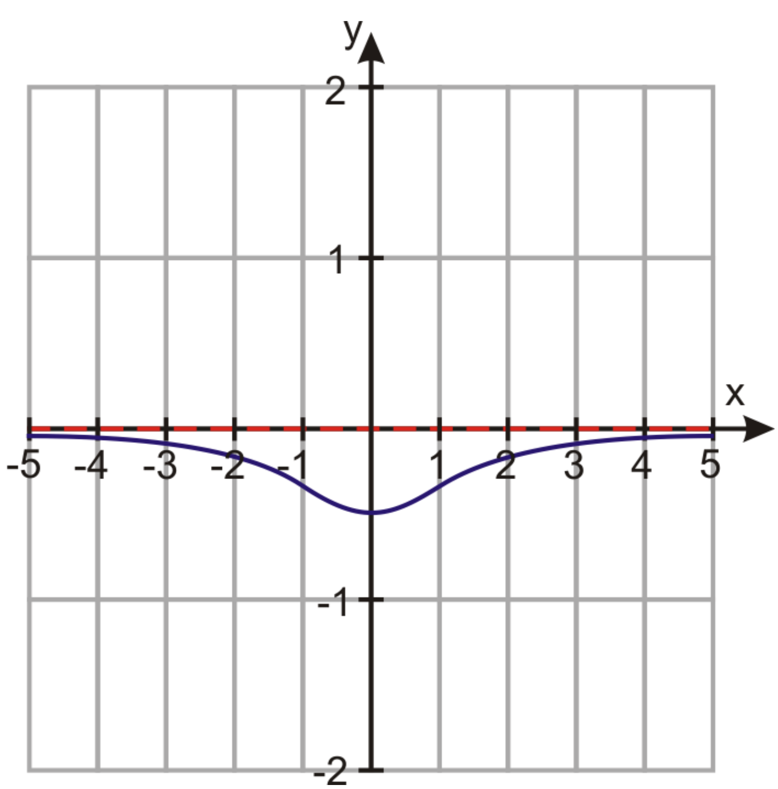


FIGURE 4.9

14.

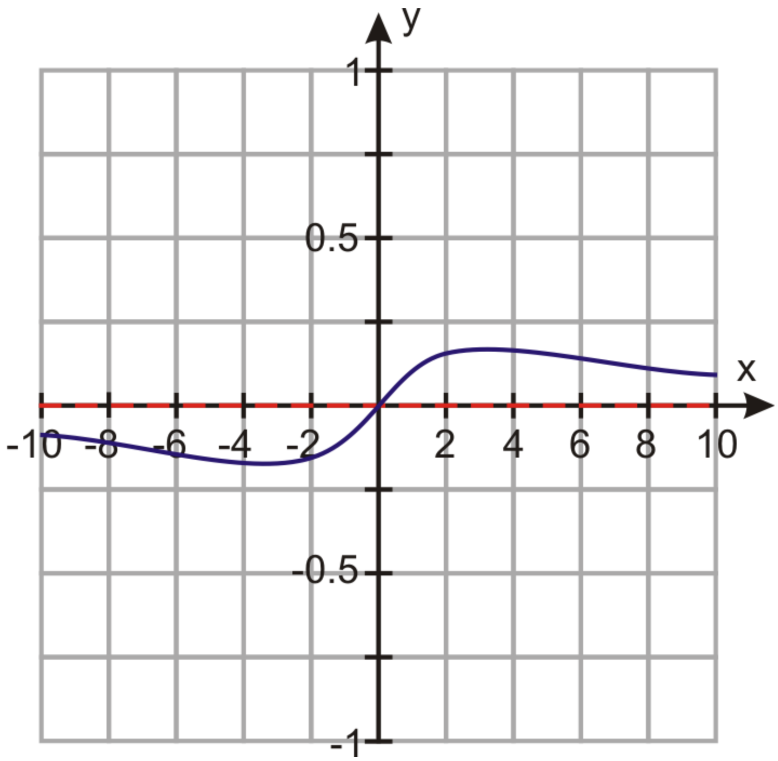


FIGURE 4.10

15.

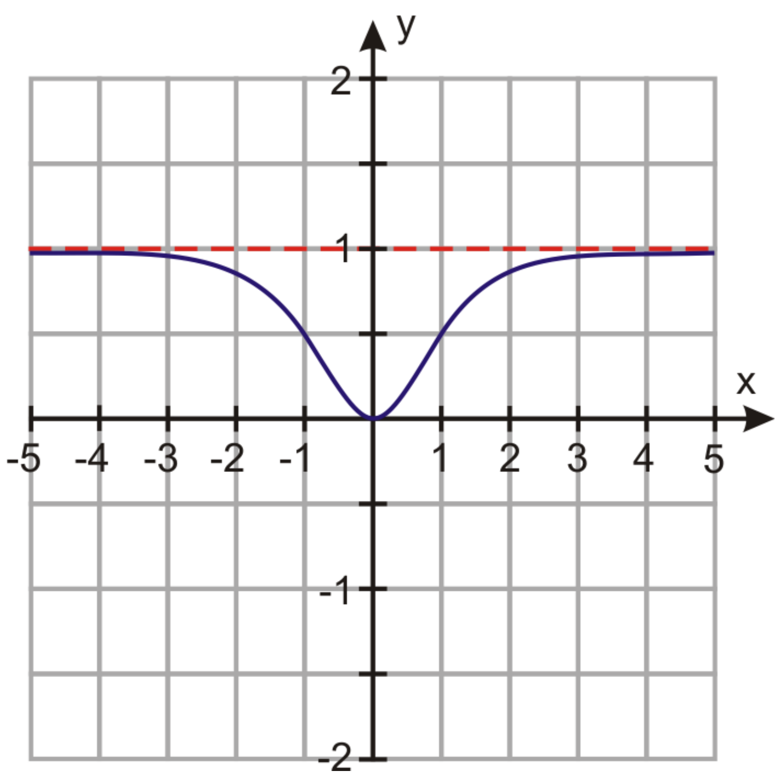


FIGURE 4.11

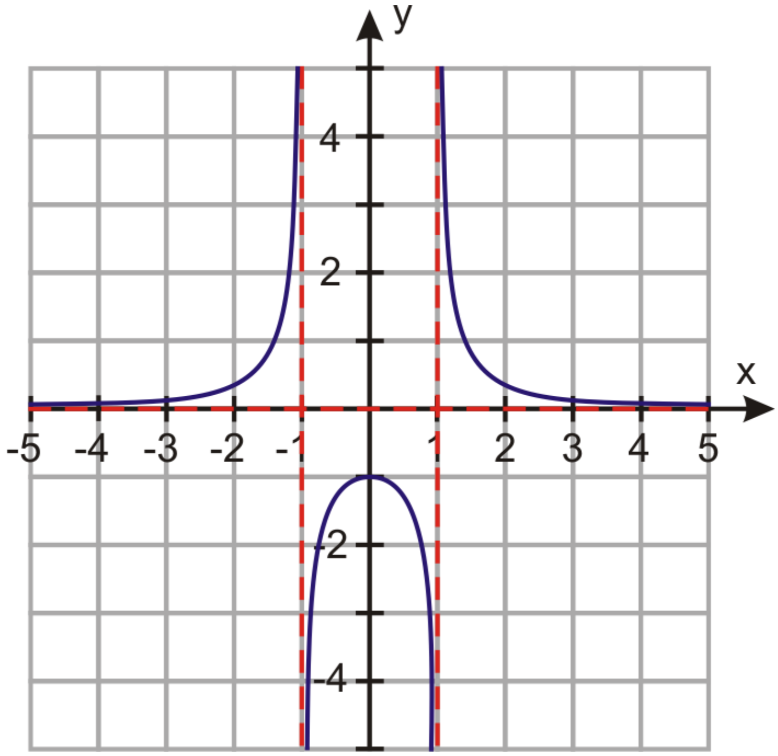
16. 

FIGURE 4.12

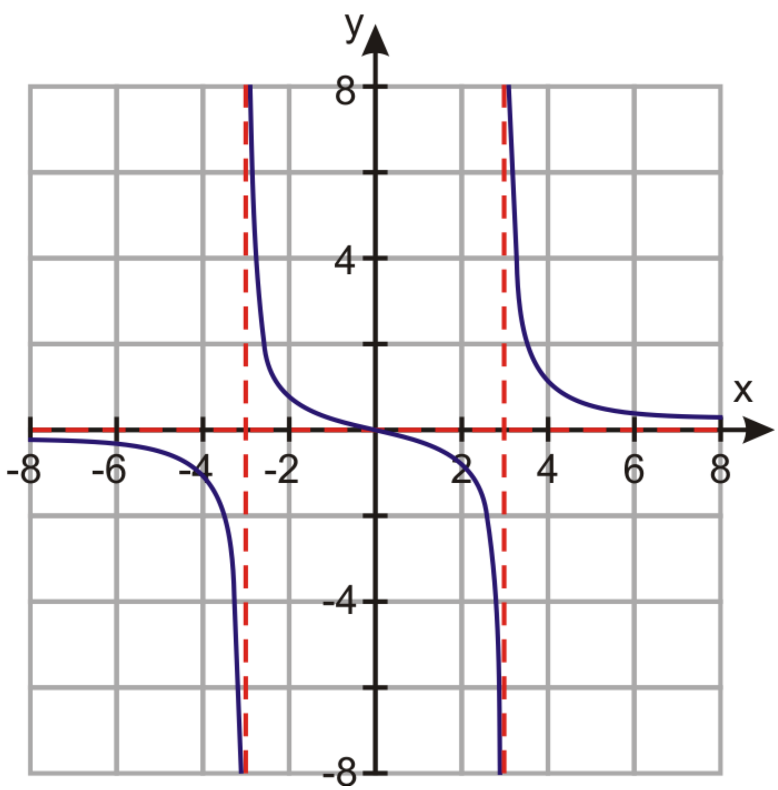
17. 

FIGURE 4.13

18.

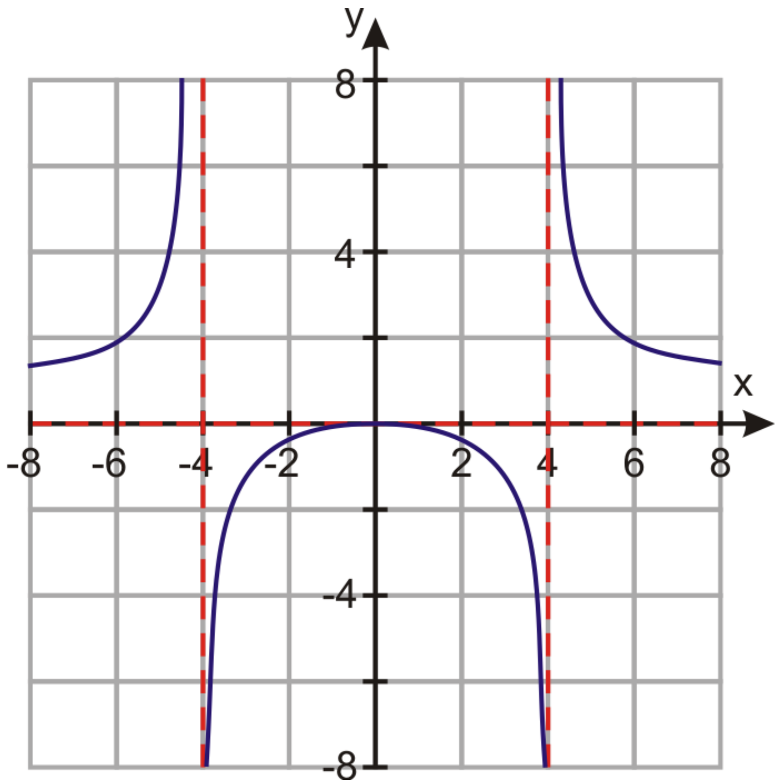


FIGURE 4.14

19.

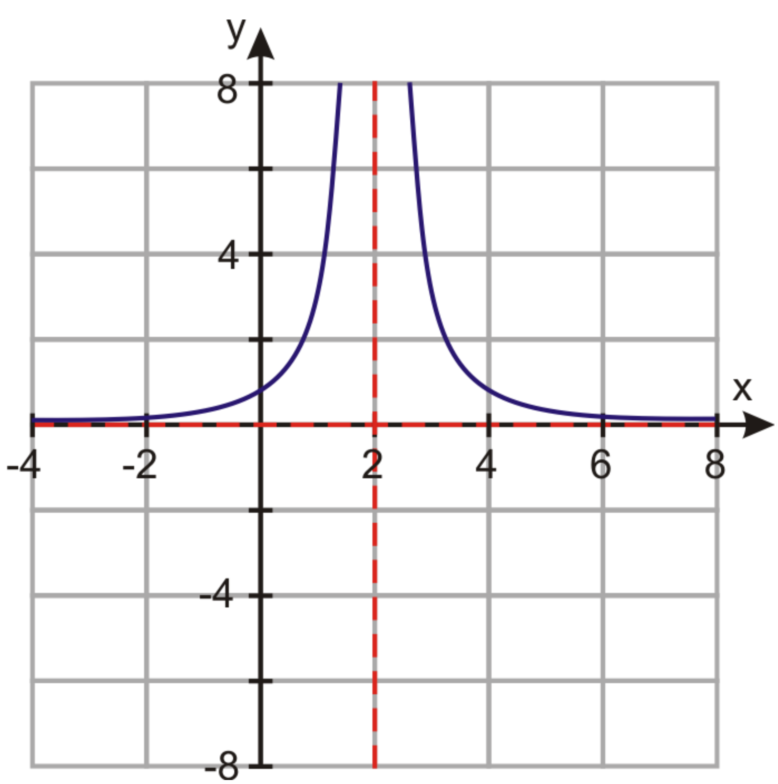


FIGURE 4.15

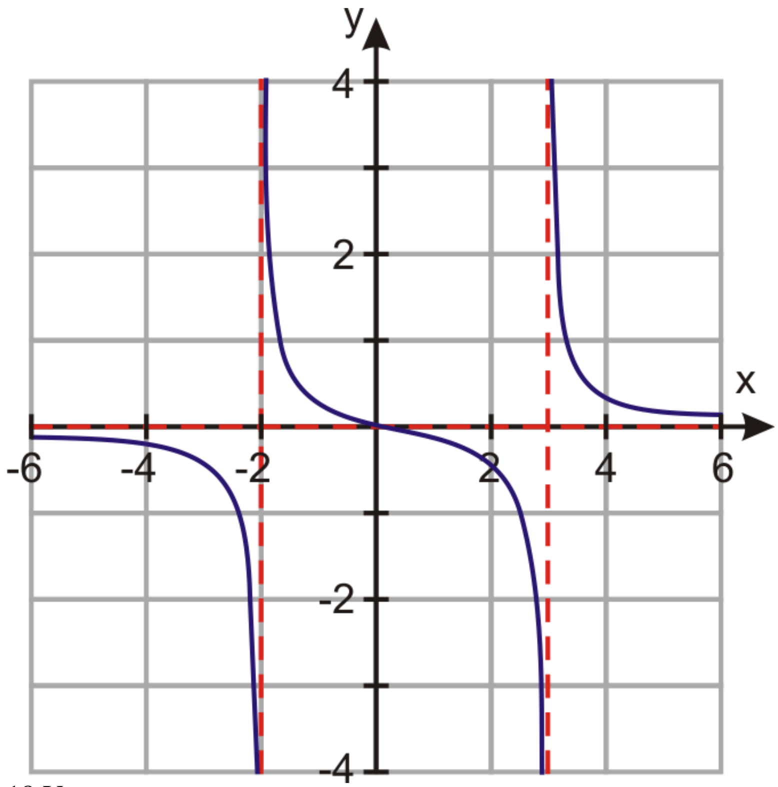
20. 

FIGURE 4.16

21. 18 V
22. 12.5 ohms
23. 2.5 Amperes
24. 10 ohms

4.3 Division of Polynomials

Learning Objectives

- Divide a polynomials by a monomial.
- Divide a polynomial by a binomial.
- Rewrite and graph rational functions.

Introduction

A **rational expression** is formed by taking the quotient of two polynomials.

Some examples of rational expressions are

a) $\frac{2x}{x^2-1}$

b) $\frac{4x^2-3x+4}{2x}$

c) $\frac{9x^2+4x-5}{x^2+5x-1}$

d) $\frac{2x^3}{2x+3}$

Just as with rational numbers, the expression on the top is called the **numerator** and the expression on the bottom is called the **denominator**. In special cases we can simplify a rational expression by dividing the numerator by the denominator.

Divide a Polynomial by a Monomial

We start by dividing a polynomial by a monomial. To do this, we divide each term of the polynomial by the monomial. When the numerator has different terms, the term on the bottom of the fraction serves as **common** denominator to all the terms in the numerator.

Example 1

Divide.

a) $\frac{8x^2-4x+16}{2}$

b) $\frac{3x^3-6x-1}{x}$

c) $\frac{-3x^2-18x+6}{9x}$

Solution

$$\frac{8x^2-4x+16}{2} = \frac{8x^2}{2} - \frac{4x}{2} + \frac{16}{2} = 4x^2-2x+8$$
$$\frac{3x^3-6x-1}{x} = \frac{3x^3}{x} + \frac{6x}{x} - \frac{1}{x} = 3x^2+6-\frac{1}{x}$$
$$\frac{-3x^2-18x+6}{9x} = \frac{3x^2}{9x} - \frac{18x}{9x} + \frac{6}{9x} = -\frac{x}{3} - 2 + \frac{2}{3x}$$

A common error is to cancel the denominator with just one term in the numerator.

Consider the quotient $\frac{3x+4}{4}$

Remember that the denominator of 4 is common to both the terms in the numerator. In other words we are dividing both of the terms in the numerator by the number 4.

The correct way to simplify is

$$\frac{3x+4}{4} = \frac{3x}{4} + \frac{4}{4} = \frac{3x}{4} + 1$$

A common mistake is to cross out the number 4 from the numerator and the denominator

$$\frac{3x+4}{4} = 3x$$

This is incorrect because the term $3x$ does not get divided by 4 as it should be.

Example 2

Divide $\frac{5x^3-10x^2+x-25}{-5x^2}$.

Solution

$$\frac{5x^3-10x^2+x-25}{-5x^2} = \frac{5x^3}{-5x^2} - \frac{10x^2}{-5x^2} + \frac{x}{-5x^2} - \frac{25}{-5x^2}$$

The negative sign in the denominator changes all the signs of the fractions:

$$-\frac{5x^3}{5x^2} + \frac{10x^2}{5x^2} - \frac{x}{5x^2} + \frac{25}{5x^2} = -x + 2 - \frac{1}{5x} + \frac{5}{x^2}$$

Divide a Polynomial by a Binomial

We divide polynomials in a similar way that we perform long division with numbers. We will explain the method by doing an example.

Example 3

Divide $\frac{x^2+4x+5}{x+3}$.

Solution: When we perform division, the expression in the numerator is called the **dividend** and the expression in the denominator is called the **divisor**.

To start the division we rewrite the problem in the following form.

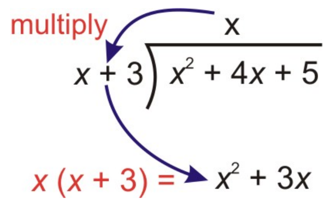
$$x+3\overline{\smash{\big)}\,x^2+4x+5}$$

We start by dividing the first term in the dividend by the first term in the divisor $\frac{x^2}{x} = x$.

We place the answer on the line above the x term.

$$\begin{array}{r} x \\ x+3\overline{)\,x^2+4x+5} \end{array}$$

Next, we multiply the x term in the answer by each of the $x+3$ in the divisor and place the result under the divided matching like terms.



Now subtract x^2+3x from x^2+4x+5. It is useful to change the signs of the terms of x^2+3x to $-x^2-3x$ and add like terms vertically.

$$\begin{array}{r} x \\ x+3\overline{)\,x^2+4x+5} \\ \underline{-x^2-3x} \\ x \end{array}$$

Now, bring down 5, the next term in the dividend.

$$\begin{array}{r} x \\ x+3\overline{)\,x^2+4x+5} \\ \underline{-x^2-3x} \\ x+5 \end{array}$$

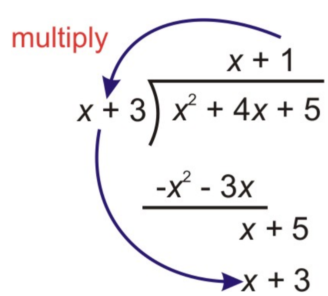
We repeat the procedure.

First divide the first term of $x+5$ by the first term of the divisor $\left(\frac{x}{x}\right)=1$.

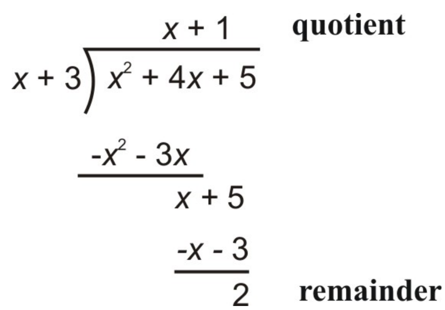
Place this answer on the line above the constant term of the dividend,

$$\begin{array}{r} x+1 \\ x+3\overline{)\,x^2+4x+5} \\ \underline{-x^2-3x} \\ x+5 \end{array}$$

Multiply 1 by the divisor $x+3$ and write the answer below $x+5$ matching like terms.



Subtract $x+3$ from $x+5$ by changing the signs of $x+3$ to $-x-3$ and adding like terms.



Since there are no more terms from the dividend to bring down, we are done.

The answer is $x+1$ with a remainder of 2.

Remember that for a division with a remainder the answer is quotient $+\frac{\text{remainder}}{\text{divisor}}$

We write our answer as.

$$\frac{x^2+4x+5}{x+3} = x+1+\frac{2}{x+3}$$

Check

To check the answer to a long division problem we use the fact that:

$$\text{divisor} \cdot \text{qiotient} + \text{remainder} = \text{divisor}$$

For the problem above here is the check of our solution.

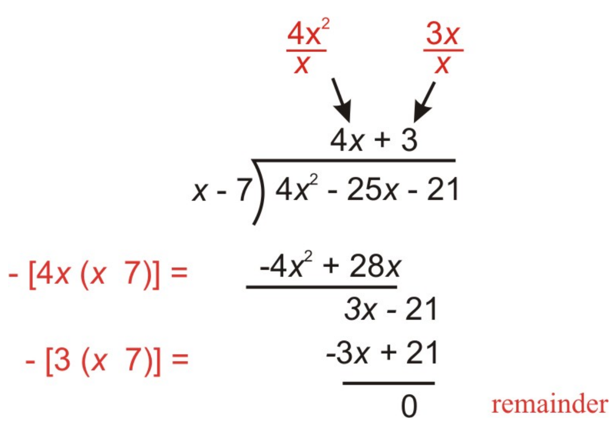
$$\begin{aligned}(x+3)(x+1)+2 &= x^2+4x+3+2 \\ &= x^2+4x+5\end{aligned}$$

The answer checks out.

Example 4

Divide $\frac{4x^2-25x-21}{x-7}$.

Solution



Answer $\frac{4x^2-25x-21}{x-7} = 4x+3$

Check $(4x+3)(x-7)+0 = 4x^2-25x-21$. The answer checks out.

Rewrite and Graph Rational Functions

In the last section we saw how to find vertical and horizontal asymptotes. Remember that the horizontal asymptote shows the value of y that the function approaches for large values of x. Lets review the method for finding horizontal asymptotes and see how it is related to polynomial division.

We can look at different types of rational functions.

Case 1 The polynomial in the numerator has a lower degree than the polynomial in the denominator. Take for example, $f(x) = \frac{2}{x-1}$

We see that we cannot divide 2 by $x-1$ and y approaches zero because the number in the denominator is bigger than the number in the numerator for large values of x.

The **horizontal asymptote is** $y = 0$.

Case 2 The polynomial in the numerator has the same degree as the polynomial in the denominator. Take for example, $f(x) = \frac{3x+2}{x-1}$

In this case, we can divide the two polynomials and obtain.

$$\begin{array}{r|l} & \quad 3 \\ \hline x-1 & 3x+2 \\ & -3x+3 \\ \hline & \quad 5 \end{array}$$

The quotient is $f(x) = 3 + \frac{5}{x-1}$.

Because the number in the denominator of the remainder is bigger than the number in the numerator of the remainder, the reminder will approach zero for large values of x leaving only the 3, thus y will approach the value of 3 for large values of x.

The horizontal asymptote is $y = 3$.

Case 3 The polynomial in the numerator has a degree that is one more than the polynomial in the denominator. Take for example, $f(x) = \frac{4x^2+3x+2}{x-1}$.

$$\begin{array}{r|l} & \quad 4x+7 \\ \hline x-1 & 4x^2+3x+2 \\ & -4x^2+4x \\ \hline & \quad 7x+2 \\ & \quad -7x+7 \\ \hline & \qquad 9 \end{array}$$

The quotient is: $y = 4x+7+\frac{9}{x-1}$.

The remainder approaches the value of zero for large values of x and the function y approaches the straight line $y = 4x+7$.When the rational function approaches a straight line for large values of x, we say that the rational function has an **oblique asymptote**. (Sometimes oblique asymptotes are also called **slant asymptotes**). The oblique asymptote is $y = 4x+7$.

Case 4 The polynomial in the numerator has a degree that in two or more than the degree in the denominator. For example, $f(x) = \frac{x^3}{x-1}$.

In this case the polynomial has no horizontal or oblique asymptotes.

Example 5

Find the horizontal or oblique asymptotes of the following rational functions.

a) $f(x) = \frac{3x^2}{x^2+4}$

b) $f(x) = \frac{x-1}{3x^2-6}$

c) $f(x) = \frac{x^4+1}{x-5}$

d) $f(x) = \frac{x^3-3x^2+4x-1}{x^2-2}$

Solution

a) We can perform the division

$$\begin{array}{r|l} & \quad 3 \\ \hline x^2+4 & 3x^2 \\ & \underline{-3x^2-12} \\ & \quad -12 \end{array}$$

The answer to the division is $y = 3 - \frac{12}{x^2+4}$

There is a horizontal asymptote at $y = 3$.

b) We cannot divide the two polynomials.

There is a horizontal asymptote at $y = 0$.

c) The power of the numerator is 3 more than the power of the denominator. There are no horizontal or oblique asymptotes.

d) We can perform the division

$$\begin{array}{r|l} & \qquad\qquad x-3 \\ \hline x^2-2 & x^3-3x^2+4x-1 \\ & \underline{-x^3 \qquad\quad +2x} \\ & \quad -3x^2+6x-1 \\ & \quad \underline{3x^2 \qquad\quad -6} \\ & \qquad\qquad 6x-7 \end{array}$$

The answer to the division is $y = x-3+\frac{6x-7}{x^2-2}$

There is an oblique asymptote at $y = x-3$.

Notice that a rational function will either have a horizontal asymptote, an oblique asymptote or neither kind. In other words horizontal or oblique asymptotes cannot exist together for the same rational function. As x gets large, y values can approach a horizontal line or an oblique line but not both. On the other hand, a rational function can have any number of vertical asymptotes at the same time that it has horizontal or oblique asymptotes.

Review Questions

Divide the following polynomials:

1. $\frac{2x+4}{2}$
2. $\frac{x-4}{x}$
3. $\frac{5x-35}{5x}$
4. $\frac{x^2+2x-5}{x}$
5. $\frac{4x^2+12x-36}{-4x}$
6. $\frac{2x^2+10x+7}{2x^2}$
7. $\frac{x^3-x}{-2x^2}$
8. $\frac{5x^4-9}{3x}$
9. $\frac{x^3-12x^2+3x-4}{12x^2}$
10. $\frac{3-6x+x^3}{-9x^3}$
11. $\frac{x^2+3x+6}{x+1}$
12. $\frac{x^2-9x+6}{x-1}$
13. $\frac{x^2+5x+4}{x+4}$
14. $\frac{x^2-10x+25}{x-5}$
15. $\frac{x^2-20x+12}{x-3}$
16. $\frac{3x^2-x+5}{x-2}$
17. $\frac{9x^2+2x-8}{x+4}$
18. $\frac{3x^2-4}{3x+1}$
19. $\frac{5x^2+2x-9}{2x-1}$
20. $\frac{x^2-6x-12}{5x+4}$

Find all asymptotes of the following rational functions:

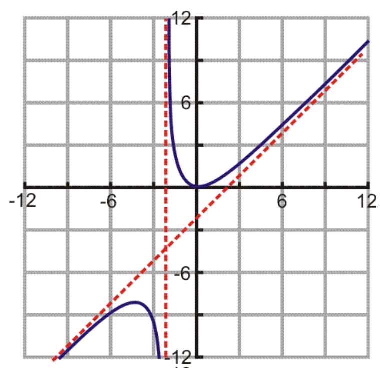
21. $f(x) = \frac{x^2}{x-2}$
22. $f(x) = \frac{1}{x+4}$
23. $f(x) = \frac{x^2-1}{x^2+1}$
24. $f(x) = \frac{x-4}{x^2-9}$
25. $f(x) = \frac{x^2+2x+1}{3x+4}$
26. $f(x) = \frac{x^3+1}{4x-1}$
27. $f(x) = \frac{x-x^3}{x^2-6x-7}$
28. $f(x) = \frac{x^4-2x}{8x+24}$

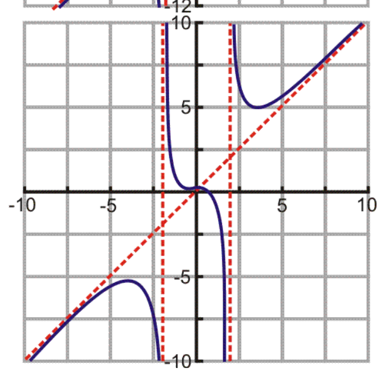
Graph the following rational functions. Indicate all asymptotes on the graph:

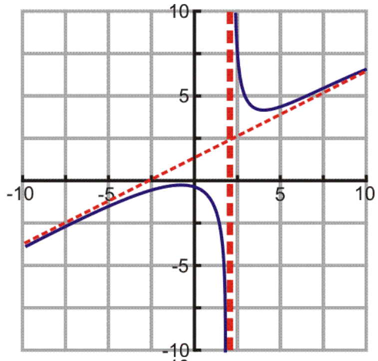
29. $f(x) = \frac{x^2}{x+2}$
30. $f(x) = \frac{x^3-1}{x^2-4}$
31. $f(x) = \frac{x^2+1}{2x-4}$
32. $f(x) = \frac{x-x^2}{3x+2}$

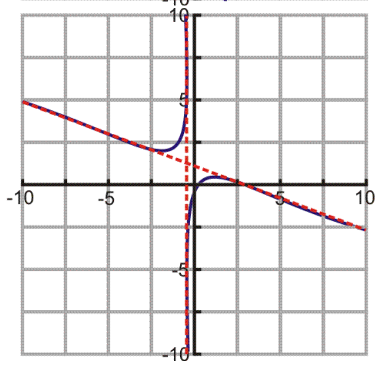
Review Answers

1. $x+2$

2. $1-\frac{4}{x}$
3. $1-\frac{7}{x}$
4. $x+2-\frac{5}{x}$
5. $-x-3+\frac{9}{x}$
6. $1+\frac{5}{x}+\frac{7}{2x^2}$
7. $-\frac{x}{2}+\frac{1}{2x}$
8. $\frac{5x^3}{3}-\frac{3}{x}$
9. $\frac{x}{12}-1+\frac{1}{4x}-\frac{1}{3x^2}$
10. $-\frac{1}{3x^3}+\frac{2}{3x^2}-\frac{1}{9}$
11. $x+2+\frac{4}{x+1}$
12. $x-8-\frac{2}{x-1}$
13. $x+1$
14. $x-5$
15. $x-17-\frac{39}{x-3}$
16. $3x+5+\frac{15}{x-2}$
17. $9x-34+\frac{128}{x+4}$
18. $x-\frac{1}{3}-\frac{11}{3(3x+1)}$
19. $\frac{5}{2}x+\frac{9}{4}-\frac{27}{4(2x-1)}$
20. $\frac{1}{5}x-\frac{34}{25}-\frac{164}{25(5x+4)}$
21. vertical: $x=2$, oblique: $y=x$
22. vertical: $x=-4$, horizontal: $y=0$
23. horizontal: $y=1$
24. vertical: $x=3, x=-3$, horizontal: $y=0$
25. vertical: $x=\frac{-4}{3}$, oblique: $y=\frac{x}{3}+\frac{2}{9}$
26. vertical: $x=\frac{1}{4}$
27. vertical: $x=7, x=-1$, oblique: $y=-x-6$
28. vertical: $x=-3$
29. 

30. 

31. 

32. 

4.4 Rational Expressions

Learning Objectives

- Simplify rational expressions.
- Find excluded values of rational expressions.
- Simplify rational models of real-world situations.

Introduction

A rational expression is reduced to lowest terms by factoring the numerator and denominator completely and canceling common factors. For example, the expression

$$\frac{x \cdot \not{z}}{y \cdot \not{z}} = \frac{x}{y}$$

simplifies to simplest form by canceling the common factor z.

Simplify Rational Expressions.

To simplify rational expressions means that the numerator and denominator of the rational expression have no common factors. In order to simplify to **lowest terms**, we factor the numerator and denominator as much as we can and cancel common factors from the numerator and the denominator of the fraction.

Example 1

Reduce each rational expression to simplest terms.

a) $\frac{4x-2}{2x^2+x-1}$

b) $\frac{x^2-2x+1}{8x-8}$

c) $\frac{x^2-4}{x^2-5x+6}$

Solution

a) Factor the numerator and denominator completely. $\frac{2(2x-1)}{(2x-1)(x+1)}$

Cancel the common term $(2x-1)$. $\frac{2}{x+1}$ **Answer**

b)Factor the numerator and denominator completely. $\frac{(x-1)(x-1)}{8(x-1)}$

Cancel the common term $(x-1)$. $\frac{x-1}{8}$ **Answer**

c) Factor the numerator and denominator completely. $\frac{(x-2)(x+2)}{(x-2)(x-3)}$

Cancel the common term $(x-2)$. $\frac{x+2}{x-3}$ **Answer**

Common mistakes in reducing fractions:

When reducing fractions, you are only allowed to cancel common **factors** from the denominator but NOT common terms. For example, in the expression

$$\frac{(x+1).(x-3)}{(x+2).(x-3)}$$

we can cross out the $(x-3)$ factor because $\frac{(x-3)}{(x-3)} = 1$.

We write

$$\frac{(x+1)\cdot\cancel{(x-3)}}{(x+2)\cdot\cancel{(x-3)}} = \frac{(x+1)}{(x+2)}$$

However, dont make the mistake of canceling out common **terms** in the numerator and denominator. For instance, in the expression.

$$\frac{x^2+1}{x^2-5}$$

we cannot cross out the x^2 terms.

$$\frac{x^2+1}{x^2-5} \neq \frac{\cancel{x^2}+1}{\cancel{x^2}-5}$$

When we cross out terms that are part of a sum or a difference we are violating the order of operations (PEMDAS). We must remember that the fraction sign means division. When we perform the operation

$$\frac{(x^2+1)}{(x^2-5)}$$

we are dividing the numerator by the denominator

$$(x^2+1) \div (x^2-5)$$

The order of operations says that we must perform the operations inside the parenthesis before we can perform the division.

Try this with numbers:

$\frac{9+1}{9-5} = \frac{10}{4} = 2.5$	But if we cancel incorrectly we obtain the following	$\frac{\cancel{9}+1}{\cancel{9}-5} = -\frac{1}{5} = -0.2.$
CORRECT	INCORRECT	

Find Excluded Values of Rational Expressions

Whenever a variable expression is present in the denominator of a fraction, we must be aware of the possibility that the denominator could be zero. Since division by zero is undefined, certain values of the variable must be **excluded**.

These values are the vertical asymptotes (i.e. values that cannot exist for x). For example, in the expression $\left(\frac{2}{x-3}\right)$, the value of $x=3$ must be excluded.

To find the excluded values we simply set the denominator equal to zero and solve the resulting equation.

Example 2

Find the excluded values of the following expressions.

a) $\frac{x}{x+4}$

b) $\frac{2x+1}{x^2-x-6}$

c) $\frac{4}{x^2-5x}$

Solution

a) When we set the denominator equal to zero we obtain. $x+4=0 \Rightarrow x=-4$ is the excluded value

b) When we set the denominator equal to zero we obtain. $x^2-x-6=0$

Solve by factoring. $(x-3)(x+2)=0$

$\Rightarrow x=3$ and $x=-2$ are the excluded values.

c) When we set the denominator equal to zero we obtain. $x^2-5x=0$

Solve by factoring. $x(x-5)=0$

$\Rightarrow x=0$ and $x=5$ are the excluded values.

Removable Zeros

Notice that in the expressions in Example 1, we removed a division by zero when we simplified the problem. For instance,

$$\frac{4x-2}{2x^2+x-1}$$

was rewritten as

$$\frac{2(2x-1)}{(2x-1)(x+1)}.$$

This expression experiences division by zero when $x=\frac{1}{2}$ and $x=-1$.

However, when we cancel common factors, we simplify the expression to $\frac{2}{x+1}$. The reduced form allows the value $x=\frac{1}{2}$. We thus removed a division by zero and the reduced expression has only $x=-1$ as the excluded value. Technically the original expression and the simplified expression are not the same. When we simplify to simplest form we should specify the removed excluded value. Thus,

$$\frac{4x-2}{2x^2+x-1}=\frac{2}{x+1}, x \neq \frac{1}{2}$$

The expression from Example 1, part *b* reduces to

$$\frac{x^2-2x+1}{8x-8}=\frac{x-1}{8}, x \neq 1$$

The expression from Example 1, part *c* reduces to

$$\frac{x^2-4}{x^2-5x+6}=\frac{x+2}{x-3}, x \neq 2$$

Simplify Rational Models of Real-World Situations

Many real world situations involve expressions that contain rational coefficients or expressions where the variable appears in the denominator.

Example 3

The gravitational force between two objects in given by the formula $F = G\frac{(m_1 m_2)}{(d^2)}$*. if the gravitation constant is given by* $G = 6.67 \times 10^{-11}$ $(N \cdot m^2/kg^2)$*. The force of attraction between the Earth and the Moon is* $F = 2.0 \times 10^{20}$ N *(with masses of* $m_1 = 5.97 \times 10^{24}$ *kg for the Earth and* $m_2 = 7.36 \times 10^{22}$ *kg for the Moon).*

What is the distance between the Earth and the Moon?

Solution

Let's start with the Law of Gravitation formula. $F = G\frac{m_1 m_2}{d^2}$

Now plug in the known values. $2.0 \times 10^{20} N = 6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2} \cdot \frac{(5.97 \times 10^{24} kg)(7.36 \times 10^{22} kg)}{d^2}$

Multiply the masses together. $2.0 \times 10^{20} N = 6.67 \times 10^{-11} \frac{N \cdot m^2}{kg^2} \cdot \frac{4.39 \times 10^{47} kg^2}{d^2}$

Cancel the kg^2 units. $2 \cdot 0 \times 10^{20} N = 6.67 \times 10^{-11} \frac{N \cdot m^2}{\cancel{kg^2}} \cdot \frac{4.39 \times 10^{47} \cancel{kg^2}}{d^2}$

Multiply the numbers in the numerator. $2.0 \times 10^{20} N = \frac{2.93 \times 10^{37}}{d^2} N \cdot m^2$

Multiply both sides by d^2. $2.0 \times 10^{20} N \cdot d^2 = \frac{2.93 \times 10^{37}}{d^2} \cdot d^2 \cdot N \cdot m^2$

Cancel common factors. $2 \cdot 0 \times 10^{20} N \cdot d^2 = \frac{2.93 \times 10^{37}}{\cancel{d^2}} \cdot \cancel{d^2} \cdot N \cdot m^2$

Simplify. $2.0 \times 10^{20} N \cdot d^2 = 2.93 \times 10^{37} N \cdot m^2$

Divide both sides by 2.0×10^{20} N. $d^2 = \frac{2.93 \times 10^{37}}{2.0 \times 10^{20}} \frac{N \cdot m^2}{N}$

Simplify. $d^2 = 1.465 \times 10^{17} m^2$

Take the square root of both sides. $d = 3.84 \times 10^8 m$ Answer

This is indeed the distance between the Earth and the Moon.

Example 4

The area of a circle is given by $A = \pi r^2$ *and the circumference of a circle is given by* $C = 2\pi r$*. Find the ratio of the circumference and area of the circle.*

Solution

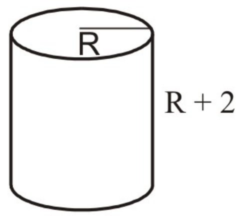
The ratio of the circumference and area of the circle is: $\frac{2\pi r}{\pi r^2}$

We cancel common factors from the numerator and denominator. $\frac{2\cancel{\pi}\cancel{R}}{\cancel{\pi}\cancel{R^2}_R}$

Simplify. $\frac{2}{r}$**Answer**

Example 5

The height of a cylinder is 2 units more than its radius. Find the ratio of the surface area of the cylinder to its volume.



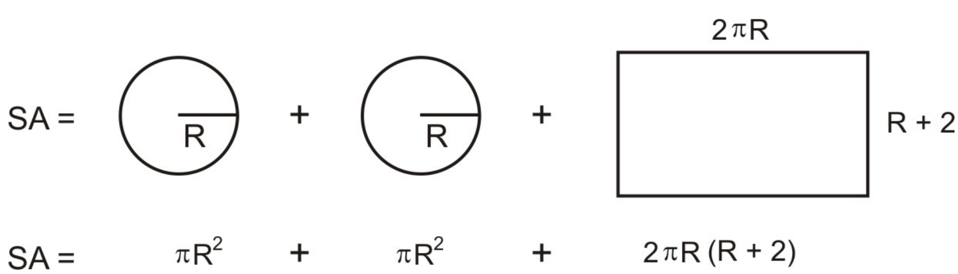
Solution

Define variables.

Let $R\ =$ the radius of the base of the cylinder.

Then, $R+2\ =$ the height of the cylinder

To find the surface area of a cylinder, we need to add the areas of the top and bottom circle and the area of the curved surface.



The volume of the cylinder is	$V=\pi R^2(R+2)$
The ratio of the surface area of the cylinder to its volume is	$\frac{2\pi R^2+2\pi R(R+2)}{\pi R^2(R+2)}$
Eliminate the parentheses in the numerator.	$\frac{2\pi R^2+2\pi R^2+4\pi R}{\pi R^2(R+2)}$
Combine like terms in the numerator.	$\frac{4\pi R^2+4\pi R}{\pi R^2(R+2)}$
Factor common terms in the numerator.	$\frac{4\pi R(R+1)}{\pi R^2(R+2)}$
Cancel common terms in the numerator and denominator.	$\frac{4\cancel{\pi}\cancel{R}(R+1)}{\cancel{\pi}\cancel{R^2}_R(R+2)}$
Simplify.	$\frac{4(R+1)}{R(R+2)}$ Answer

Review Questions

Reduce each fraction to lowest terms.

1. $\frac{4}{2x-8}$
2. $\frac{x^2+2x}{x}$
3. $\frac{9x+3}{12x+4}$
4. $\frac{6x^2+2x}{4x}$
5. $\frac{x-2}{x^2-4x+4}$
6. $\frac{x^2-9}{5x+15}$
7. $\frac{x^2+6x+8}{x^2+4x}$
8. $\frac{2x^2+10x}{x^2+10x+25}$
9. $\frac{x^2+6x+5}{x^2-x-2}$
10. $\frac{x^2-16}{x^2+2x-8}$
11. $\frac{3x^2+3x-18}{2x^2+5x-3}$
12. $\frac{x^3+x^2-20x}{6x^2+6x-120}$

Find the excluded values for each rational expression.

13. $\frac{2}{x}$
14. $\frac{4}{x+2}$
15. $\frac{2x-1}{(x-1)^2}$
16. $\frac{3x+1}{x^2-4}$
17. $\frac{x^2}{x^2+9}$
18. $\frac{2x^2+3x-1}{x^2-3x-28}$
19. $\frac{5x^3-4}{x^2+3x}$
20. $\frac{9}{x^3+11x^2+30x}$
21. $\frac{4x-1}{x^2+3x-5}$
22. $\frac{5x+11}{3x^2-2x-4}$
23. $\frac{x^2-1}{2x^2+x+3}$
24. $\frac{12}{x^2+6x+1}$
25. In an electrical circuit with resistors placed in parallel, the reciprocal of the total resistance is equal to the sum of the reciprocals of each resistance. $\frac{1}{R_c} = \frac{1}{R_1} + \frac{1}{R_2}$. If $R_1 = 25\ \Omega$ and the total resistance is $R_c = 10\ \Omega$, what is the resistance R_2?
26. Suppose that two objects attract each other with a gravitational force of 20 Newtons. If the distance between the two objects is doubled, what is the new force of attraction between the two objects?
27. Suppose that two objects attract each other with a gravitational force of 36 Newtons. If the mass of both objects was doubled, and if the distance between the objects was doubled, then what would be the new force of attraction between the two objects?
28. A sphere with radius r has a volume of $\frac{4}{3}\pi r^3$ and a surface area of $4\pi r^2$. Find the ratio the surface area to the volume of a sphere.
29. The side of a cube is increased by a factor of two. Find the ratio of the old volume to the new volume.
30. The radius of a sphere is decreased by four units. Find the ratio of the old volume to the new volume.

Review Answers

1. $\frac{2}{x-4}$
2. $x+2, x \neq 0$
3. $\frac{3}{4}, x \neq -\frac{1}{3}$
4. $\frac{3x+1}{2}, x \neq 0$
5. $\frac{1}{x-2}$

6. $\frac{x-3}{5}, x \neq -3$
7. $\frac{x+2}{x}, x \neq -4$
8. $\frac{2x}{x+5}$
9. $\frac{x+5}{x-2}, x \neq -1$
10. $\frac{x-4}{x-2}, x \neq -4$
11. $\frac{3x-6}{2x-1}, x \neq -3$
12. .
13. $x = 0$
14. $x = -2$
15. $x = 1$
16. $x = 2, x = -2$
17. none
18. $x = -4, x = 7$
19. $x = 0, x = -3$
20. $x = 0, x = -5, x = -6$
21. $x = 1.19, x = -4.19$
22. $x = 1.54, x = -0.87$
23. none
24. $x = -0.17, x = -5.83$
25. $R_c = 16\frac{2}{3}\ \Omega$
26. 5 Newtons
27. 36 Newtons
28. $\frac{3}{R}$
29. $\frac{1}{8}$
30. $\frac{R^3}{(R-4)^3}$

4.5 Multiplication and Division of Rational Expressions

Learning Objectives

- Multiply rational expressions involving monomials.
- Multiply rational expressions involving polynomials.
- Multiply a rational expression by a polynomial.
- Divide rational expressions involving polynomials.
- Divide a rational expression by a polynomial.
- Solve real-world problems involving multiplication and division of rational. expressions

Introduction

The rules for multiplying and dividing rational expressions are the same as the rules for multiplying and dividing rational numbers. Lets start by reviewing multiplication and division of fractions. When we multiply two fractions we multiply the numerators and denominators separately

$$\frac{a}{b}\cdot\frac{c}{d}=\frac{a\cdot c}{b\cdot d}$$

When we divide two fractions we first change the operation to multiplication. Remember that division is the reciprocal operation of multiplication or you can think that division is the same as multiplication by the reciprocal of the number.

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{c}$$

The problem is completed by multiplying the numerators and denominators separately $\frac{a\cdot d}{b\cdot c}$.

Multiply Rational Expressions Involving Monomials

Example 1

Multiply $\frac{4}{5}\cdot\frac{15}{8}$.

Solution

We follow the multiplication rule and multiply the numerators and the denominators separately.

$$\frac{4}{5}\cdot\frac{15}{8}=\frac{4\cdot 15}{5\cdot 8}=\frac{60}{40}$$

Notice that the answer is not in simplest form. We can cancel a common factor of 20 from the numerator and denominator of the answer.

$$\frac{60}{40} = \frac{3}{2}$$

We could have obtained the same answer a different way: by reducing common factors *before* multiplying.

$$\frac{4}{5} \cdot \frac{15}{8} = \frac{4 \cdot 15}{5 \cdot 8}$$

We can cancel a factor of 4 from the numerator and denominator:

$$\frac{4}{5} \cdot \frac{15}{8} = \frac{\cancel{4}^{1} \cdot 15}{5 \cdot \cancel{8}_{2}}$$

We can also cancel a factor of 5 from the numerator and denominator:

$$\frac{1}{5} \cdot \frac{15}{2} = \frac{1 \cdot \cancel{15}^{3}}{\cancel{5}_{1} \cdot 2} = \frac{1 \cdot 3}{1 \cdot 2} = \frac{3}{2}$$

Answer The final answer is $\frac{3}{2}$, no matter which you you go to arrive at it.

Multiplying rational expressions follows the same procedure.

- Cancel common factors from the numerators and denominators of the fractions.
- Multiply the leftover factors in the numerator and denominator.

Example 2

Multiply the following $\frac{a}{16b^8} \cdot \frac{4b^3}{5a^2}$.

Solution

Cancel common factors from the numerator and denominator.

$$\frac{\cancel{a}^{1}}{\cancel{16}_{4} \cdot \cancel{b^8}_{b^5}} \cdot \frac{\cancel{4}^{1} \cdot \cancel{b^3}^{1}}{\cancel{5a^2}_{a}}$$

When we multiply the left-over factors, we get

$$\frac{1}{4ab^5} \text{ Answer}$$

Example 3

Multiply $9x^2 \cdot \frac{4y^2}{21x^4}$.

Solution

Rewrite the problem as a product of two fractions.

$$\frac{9x^2}{1} \cdot \frac{4y^2}{21x^4}$$

Cancel common factors from the numerator and denominator

$$\frac{\cancel{9}^3\cancel{x^2}}{1} \cdot \frac{4y^2}{\cancel{21}_7\cancel{x^4}_{x^2}}$$

We multiply the left-over factors and get

$$\frac{12y^2}{7x^2} \text{ Answer}$$

Multiply Rational Expressions Involving Polynomials

When multiplying rational expressions involving polynomials, the first step involves factoring all polynomials expressions as much as we can. We then follow the same procedure as before.

Example 4

Multiply $\frac{4x+12}{3x^2} \cdot \frac{x}{x^2-9}$.

Solution

Factor all polynomial expression when possible.

$$\frac{4(x+3)}{3x^2} \cdot \frac{x}{(x+3)(x-3)}$$

Cancel common factors in the numerator and denominator of the fractions:

$$\frac{4\cancel{(x+3)}}{3\cancel{x^2}_x} \cdot \frac{\cancel{x}}{\cancel{(x+3)}(x-3)}$$

Multiply the left-over factors.

$$\frac{4}{3x(x-3)} = \frac{4}{3x^2-9x} \text{ Answer}$$

Example 5

Multiply $\frac{12x^2-x-6}{x^2-1} \cdot \frac{x^2+7x+6}{4x^2-27x+18}$.

Solution

Factor all polynomial expression when possible.

$$\frac{(3x+2)(4x-3)}{(x+1)(x-1)} \cdot \frac{(x+1)(x+6)}{(4x-3)(x-6)}$$

Cancel common factors in the numerator and denominator of the fractions.

$$\frac{(3x+2)\cancel{(4x-3)}}{\cancel{(x+1)}(x-1)} \cdot \frac{\cancel{(x+1)}(x+6)}{\cancel{(4x-3)}(x-6)}$$

Multiply the remaining factors.

$$\frac{(3x+2)(x+6)}{(x-1)(x-6)} = \frac{3x^2+20x+12}{x^2-7x+6} \text{ Answer}$$

Multiply a Rational Expression by a Polynomial

When we multiply a rational expression by a whole number or a polynomial, we must remember that we can write the whole number (or polynomial) as a fraction with denominator equal to one. We then proceed the same way as in the previous examples.

Example 6

Multiply $\frac{3x+18}{4x^2+19x-5} \cdot x^2+3x-10$.

Solution

Rewrite the expression as a product of fractions.

$$\frac{3x+18}{4x^{+}19x-5} \cdot \frac{x^2+3x-10}{1}$$

Factor all polynomials possible and cancel common factors.

$$\frac{3x(x+6)}{\cancel{(x+5)}(4x-1)} \cdot \frac{(x-2)\cancel{(x+5)}}{1}$$

Multiply the remaining factors.

$$\frac{(3x+18)(x-2)}{4x-1} = \frac{3x^2+12x-36}{4x-1}$$

Divide Rational Expressions Involving Polynomials

Since division is the reciprocal of the multiplication operation, we first rewrite the division problem as a multiplication problem and then proceed with the multiplication as outlined in the previous example.

Note: Remember that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$. The first fraction remains the same and you take the reciprical of the *second* fraction. Do not fall in the common trap of flipping the first fraction.

Example 7

Divide $\frac{4x^2}{15} \div \frac{6x}{5}$.

Solution

First convert into a multiplication problem by flipping what we are dividing by and then simplify as usual.

$$\frac{\cancel{4}^2\cancel{x^2}^x}{\cancel{15}} \cdot \frac{\cancel{5}^2}{\cancel{6x}_1} = \frac{2\cdot x\cdot 1}{3\cdot 3\cdot 1} = \frac{2x}{9}$$

Example 8

Divide $\frac{3x^2-15x}{2x^2+3x-14} \div \frac{x^2-25}{2x^2+13x+21}$.

Solution

First convert into a multiplication problem by flipping what we are dividing by and then simplify as usual.

$$\frac{3x^2-15x}{2x^2+3x-14} \cdot \frac{2x^2+13x+21}{x^2-25}$$

Factor all polynomials and cancel common factors.

$$\frac{3x\cancel{(x-5)}}{\cancel{(2x+7)}(x-2)} \cdot \frac{\cancel{(2x+7)}(x-2)}{\cancel{(x-5)}(x+5)}$$

Multiply the remaining factors.

$$\frac{3x(x+3)}{(x-2)(x+5)} = \frac{3x^2+9x}{x^2+3x-10} \text{ Answer.}$$

Divide a Rational Expression by a Polynomial

When we divide a rational expression by a whole number or a polynomial, we must remember that we can write the whole number (or polynomial) as a fraction with denominator equal to one. We then proceed the same way as in the previous examples.

Example 9

Divide $\frac{9x^2-4}{2x-2} \div 21x^2-2x-8$.

Solution

Rewrite the expression as a division of fractions.

$$\frac{9x^2-4}{2x-2} \div \frac{21x^2-2x-8}{1}$$

Convert into a multiplication problem by taking the reciprocal of the divisor (i.e. what we are dividing by).

$$\frac{9x^2-4}{2x-2} \cdot \frac{1}{21x^2-2x-8}$$

Factor all polynomials and cancel common factors.

$$\frac{\cancel{(3x-2)}(3x+2)}{2(x-1)} \cdot \frac{1}{\cancel{(3x-2)}(7x+4)}$$

Multiply the remaining factors.

$$\frac{3x+2}{14x^2-6x-8}.$$

Solve Real-World Problems Involving Multiplication and Division of Rational Expressions

Example 10

Suppose Marciel is training for a running race. Marciels speed (in miles per hour) of his training run each morning is given by the function x^3-9x*, where x is the number of bowls of cereal he had for breakfast* $(1 \le x \le 6)$*. Marciels training distance (in miles), if he eats x bowls of cereal, is* $3x^2-9x$*. What is the function for Marciels time and how long does it take Marciel to do his training run if he eats five bowls of cereal on Tuesday morning?*

Solution

$$\begin{aligned}
\text{time} &= \frac{\text{distance}}{\text{speed}} \\
\text{time} &= \frac{3x^2-9x}{x^3-9x} = \frac{3x(x-3)}{x(x^2-9)} = \frac{3x\cancel{(x-3)}}{x(x+3)\cancel{(x-3)}} \\
\text{time} &= \frac{3}{x+3}
\end{aligned}$$

If $x=5$, then

$$\text{time} = \frac{3}{5+3} = \frac{3}{8}$$

Answer Marciel will run for $\frac{3}{8}$ of an hour.

Review Questions

Perform the indicated operation and reduce the answer to lowest terms

1. $\frac{x^3}{2y^3} \cdot \frac{2y^2}{x}$
2. $2xy \div \frac{2x^2}{y}$
3. $\frac{2x}{y^2} \cdot \frac{4y}{5x}$
4. $2xy \cdot \frac{2y^2}{x^3}$
5. $\frac{4y^2-1}{y^2-9} \cdot \frac{y-3}{2y-1}$
6. $\frac{6ab}{a^2} \cdot \frac{a^3b}{3b^2}$
7. $\frac{x^2}{x-1} \div \frac{x}{x^2+x-2}$
8. $\frac{33a^2}{-5} \cdot \frac{20}{11a^3}$

9. $\frac{a^2+2ab+b^2}{ab^2-a^2b} \div (a+b)$
10. $\frac{2x^2+2x-24}{x^2+3x} \cdot \frac{x^2+x-6}{x+4}$
11. $\frac{3-x}{3x-5} \div \frac{x^2-9}{2x^2-8x-10}$
12. $\frac{x^2-25}{x+3} \div (x-5)$
13. $\frac{2x+1}{2x-1} \div \frac{4x^2-1}{1-2x}$
14. $\frac{x}{x-5} \cdot \frac{x^2-8x+15}{x^2-3x}$
15. $\frac{3x^2+5x-12}{x^2-9} \div \frac{3x-4}{3x+4}$
16. $\frac{5x^2+16x+3}{36x^2-25} \cdot (6x^2+5x)$
17. $\frac{x^2+7x+10}{x^2-9} \cdot \frac{x^2-3x}{3x^2+4x-4}$
18. $\frac{x^2+x-12}{x^2+4x+4} \div \frac{x-3}{x+2}$
19. $\frac{x^4-16}{x^2-9} \div \frac{x^2+4}{x^2+6x+9}$
20. $\frac{x^2+8x+16}{7x^2+9x+2} \div \frac{7x+2}{x^2+4x}$
21. Marias recipe asks for $2\frac{1}{2}$ times more flour than sugar. How many cups of flour should she mix in if she uses $3\frac{1}{3}$ cups of sugar?
22. George drives from San Diego to Los Angeles. On the return trip, he increases his driving speed by 15 miles per hour. In terms of his initial speed, by what factor is the driving time decreased on the return trip?
23. Ohms Law states that in an electrical circuit $I = \frac{V}{R_c}$. The total resistance for resistors placed in parallel is given by $\frac{1}{R_{tot}} = \frac{1}{R_1} + \frac{1}{R_2}$. Write the formula for the electric current in term of the component resistances: R_1 and R_2.

Review Answers

1. $\frac{x^2}{y}$
2. $\frac{y^2}{x}$
3. $\frac{8}{5y}$
4. $\frac{47^3}{x^2}$
5. $\frac{2y+1}{y+3}$
6. $2a^2$
7. x^2+2x
8. $\frac{-12}{a}$
9. $\frac{a+b}{ab^2-a^2b}$
10. $\frac{2x^2-10x+12}{x}$
11. $\frac{-2x^2+8x+10}{3x^2+4x-15}$
12. $\frac{x+5}{x+3}$
13. $\frac{1}{1-2x}$
14. 1
15. $\frac{3x+4}{x-3}$
16. $\frac{5x^3+16x^2+3x}{(6x-5)}$
17. $\frac{x^2+5x}{3x^2-11x+6}$
18. $\frac{x+4}{x+2}$
19. $\frac{x^2-4}{x^2-9}$
20. $\frac{x+4}{x^2+x}$
21. $8\frac{1}{3}$ cups
22. $\frac{s}{s+15}$
23. $I = \frac{E}{R_1} + \frac{E}{R_2}$

4.6 Addition and Subtraction of Rational Expressions

Learning Objectives

- Add and subtract rational expressions with the same denominator.
- Find the least common denominator of rational expressions.
- Add and subtract rational expressions with different denominators.
- Solve real-world problems involving addition and subtraction of rational expressions.

Introduction

Like fractions, rational expressions represent a portion of a quantity. Remember that when we add or subtract fractions we must first make sure that they have the same denominator. Once the fractions have the same denominator, we combine the different portions by adding or subtracting the numerators and writing that answer over the common denominator.

Add and Subtract Rational Expressions with the Same Denominator

Fractions with common denominators combine in the following manner.

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c} \qquad \text{and} \qquad \frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$$

Example 1

Simplify.

a) $\frac{8}{7}-\frac{2}{7}+\frac{4}{7}$

b) $\frac{4x^2-3}{x+5}+\frac{2x^2-1}{x+5}$

c) $\frac{x^2-2x+1}{2x+3}-\frac{3x^2-3x+5}{2x+3}$

Solution

a) Since the denominators are the same we combine the numerators.

$$\frac{8}{7}-\frac{2}{7}+\frac{4}{7}=\frac{8-2+4}{7}=\frac{10}{7} \text{ Answer}$$

b) Since the denominators are the same we combine the numerators.

$$\frac{4x^2-3+2x^2-1}{x+5}$$

Simplify by collecting like terms.

$$\frac{6x^2-4}{x+5} \text{ Answer}$$

c) Since the denominators are the same we combine the numerators. Make sure the subtraction sign is distributed to all terms in the second expression.

$$\begin{aligned}
&\frac{x^2-2x+1-(3x^2-3x+5)}{2x+3}\\
&=\frac{x^2-2x+1-3x^2+3x-5}{2x+3}\\
&=\frac{-2x^2+x-4}{2x+3} \text{ Answer}
\end{aligned}$$

Find the Least common Denominator of Rational Expressions

To add and subtract fractions with different denominators, we must first rewrite all fractions so that they have the same denominator. In general, we want to find the **least common denominator.** To find the least common denominator, we find the **least common multiple** (LCM) of the expressions in the denominators of the different fractions. Remember that the least common multiple of two or more integers is the least positive integer having each as a factor.

Consider the integers 234, 126 and 273.

To find the least common multiple of these numbers we write each integer as a product of its prime factors.

Here we present a systematic way to find the prime factorization of a number.

- Try the prime numbers, in order, as factors.
- Use repeatedly until it is no longer a factor.
- Then try the next prime:

$$\begin{aligned}
234 &= 2\cdot 117\\
&= 2\cdot 3\cdot 39\\
&= 2\cdot 3\cdot 3\cdot 13\\
234 &= 2\cdot 3^2\cdot 13
\end{aligned}$$

$$\begin{aligned}
126 &= 2\cdot 63\\
&= 2\cdot 3\cdot 21\\
&= 2\cdot 3\cdot 3\cdot 7\\
126 &= 2\cdot 3^2\cdot 7
\end{aligned}$$

$$\begin{aligned}
273 &= 3\cdot 91\\
&= 3\cdot 7\cdot 13
\end{aligned}$$

Once we have the prime factorization of each number, the least common multiple of the numbers is the product of all the different factors taken to the highest power that they appear in any of the prime factorizations. In this case, the factor of two appears at most once, the factor of three appears at most twice, the factor of seven appears at most once, the factor of 13 appears at most once. Therefore,

$$\text{LCM} = 2 \cdot 3^2 \cdot 7 \cdot 13 = 1638 \text{ Answer}$$

If we have integers that have no common factors, the least common multiple is just the product of the integers. Consider the integers 12 and 25.

$$12 = 2^2 \cdot 3 \qquad \text{and} \qquad 25 = 5^2$$

The $\text{LCM} = 2^2 \cdot 3 \cdot 5^2 = 300$, which is just the product of 12 and 25.

The procedure for finding the lowest common multiple of polynomials is similar. We rewrite each polynomial in factored form and we form the LCM by taken each factor to the highest power it appears in any of the separate expressions.

Example 2

Find the LCM of $48x^2y$ *and* $60xy^3z$.

Solution

First rewrite the integers in their prime factorization.

$$48 = 2^4 \cdot 3$$
$$60 = 2^2 \cdot 3 \cdot 5$$

Therefore, the two expressions can be written as

$$48x^2y = 2^4 \cdot 3 \cdot x^2 \cdot y$$
$$60xy^3z = 2^2 \cdot 3 \cdot 5 \cdot x \cdot y^3 \cdot z$$

The LCM is found by taking each factor to the highest power that it appears in either expression.

$$\text{LCM} = 2^4 \cdot 3 \cdot 5 \cdot x^2 \cdot y^3 \cdot z = 240x^2y^3z.$$

Example 3

Find the LCM of $2x^2 + 8x + 8$ *and* $x^3 - 4x^2 - 12x$.

Solution

Factor the polynomials completely.

$$2x^2 + 8x + 8 = 2(x^2 + 4x + 4) = 2(x + 2)^2$$
$$x^3 - 4x^2 - 12x = x(x^2 - 4x - 12) = x(x - 6)(x + 2)$$

The LCM is found by taking each factor to the highest power that it appears in either expression.

$$\text{LCM} = 2x(x+2)^2(x-6) \text{ Answer}$$

It is customary to leave the LCM in factored form because this form is useful in simplifying rational expressions and finding any excluded values.

Example 4

Find the LCM of x^2-25 *and* x^2+3x+2.

Solution

Factor the polynomials completely:

$$\begin{aligned} x^2-25 &= (x+5)(x-5) \\ x^2+3x+2 &= (x+2)(x+1) \end{aligned}$$

Since the two expressions have no common factors, the LCM is just the product of the two expressions.

$$\text{LCM} = (x+5)(x-5)(x+2)(x+1) \text{ Answer}$$

Add and Subtract Rational Expressions with Different Denominators

Now we are ready to add and subtract rational expressions. We use the following procedure.

1. Find the **least common denominator** (LCD) of the fractions.
2. Express each fraction as an equivalent fraction with the LCD as the denominator.
3. Add or subtract and simplify the result.

Example 5

Add $\frac{4}{12}+\frac{5}{18}$.

Solution

We can write the denominators in their prime factorization $12 = 2^2 \cdot 3$ and $18 = 2 \cdot 3^2$. The least common denominator of the fractions is the LCM of the two numbers: $2^2 \cdot 3^2 = 36$. Now we need to rewrite each fraction as an equivalent fraction with the LCD as the denominator.

For the first fraction. 12 needs to be multiplied by a factor of 3 in order to change it into the LCD, so we multiply the numerator and the denominator by 3.

$$\frac{4}{12} \cdot \frac{3}{3} = \frac{12}{36}$$

For the section fraction. 18 needs to be multiplied by a factor of 2 in order to change it into the LCD, so we multiply the numerator and the denominator by 2.

$$\frac{5}{18} \cdot \frac{2}{2} = \frac{10}{36}$$

Once the denominators of the two fractions are the same we can add the numerators.

$$\frac{12}{36}+\frac{10}{36}=\frac{22}{36}$$

The answer can be reduced by canceling a common factor of 2.

$$\frac{12}{36}+\frac{10}{36}=\frac{22}{36}=\frac{11}{18} \text{ Answer}$$

Example 6

Perform the following operation and simplify.

$$\frac{2}{x+2}-\frac{3}{2x-5}$$

Solution

The denominators cannot be factored any further, so the LCD is just the product of the separate denominators.

$$\text{LCD}=(x+2)(2x-5)$$

The first fraction needs to be multiplied by the factor $(2x-5)$ and the second fraction needs to be multiplied by the factor $(x+2)$.

$$\frac{2}{x+2}\cdot\frac{(2x-5)}{(2x-5)}-\frac{3}{2x-5}\cdot\frac{(x+2)}{(x+2)}$$

We combine the numerators and simplify.

$$\frac{2(2x-5)-3(x+2)}{(x+2)(2x-5)}=\frac{4x-10-3x-6}{(x+2)(2x-5)}$$

Combine like terms in the numerator.

$$\frac{x-16}{(x+2)(2x-5)} \text{ Answer}$$

Example 8

Perform the following operation and simplify.

$$\frac{4x}{x-5}-\frac{3x}{5-x}$$

Solution

Notice that the denominators are almost the same. They differ by a factor of -1.

Factor $(a-1)$ from the second denominator.

$$\frac{4x}{x-5}-\frac{3x}{-(x-5)}$$

The two negative signs in the second fraction cancel.

$$\frac{4x}{x-5}+\frac{3x}{(x-5)}$$

Since the denominators are the same we combine the numerators.

$$\frac{7x}{x-5} \text{ Answer}$$

Example 9

Perform the following operation and simplify.

$$\frac{2x-1}{x^2-6x+9}-\frac{3x+4}{x^2-9}$$

Solution

We factor the denominators.

$$\frac{2x-1}{(x-3)^2}-\frac{3x+4}{(x+3)(x-3)}$$

The LCD is the product of all the different factors taken to the highest power they have in either denominator. $\text{LCD}=(x-3)^2(x+3)$.

The first fraction needs to be multiplied by a factor of $(x+3)$ and the second fraction needs to be multiplied by a factor of $(x-3)$.

$$\frac{2x-1}{(x-3)^2}\cdot\frac{(x+3)}{(x+3)}-\frac{3x+4}{(x+3)(x-3)}\cdot\frac{(x-3)}{(x-3)}$$

Combine the numerators.

$$\frac{(2x-1)(x+3)-(3x+4)(x-3)}{(x-3)^2(x+3)}$$

Eliminate all parentheses in the numerator.

$$\frac{2x^2+5x-3-(3x^2-5x-12)}{(x-3)^2(x+3)}$$

Distribute the negative sign in the second parenthesis.

$$\frac{2x^2+5x-3-3x^2+5x+12}{(x-3)^2(x+3)}$$

Combine like terms in the numerator.

$$\frac{-x^2+10x+9}{(x-3)^2(x+3)} \text{ Answer}$$

Solve Real-World Problems Involving Addition and Subtraction of Rational Expressions

Example 9

In an electrical circuit with two resistors placed in parallel, the reciprocal of the total resistance is equal to the sum of the reciprocals of each resistance $\frac{1}{R_{tot}} = \frac{1}{R_1} + \frac{1}{R_2}$. *Find an expression for the total resistance in a circuit with two resistors wired in parallel.*

Solution

The expression for the relationship between total resistance and resistances placed in parallel says that the reciprocal of the total resistance is the sum of the reciprocals of the individual resistances.

Lets simplify the expression $\frac{1}{R_1} + \frac{1}{R_2}$.

The lowest common denominator is

$$= R_1R_2$$

Multiply the first fraction by $\frac{R_2}{R_2}$ and the second fraction by $\frac{R_1}{R_1}$.

$$\frac{R_2}{R_2} \cdot \frac{1}{R_1} + \frac{R_1}{R_1} \cdot \frac{1}{R_2}$$

Simplify.

$$\frac{R_2+R_1}{R_1R_2}$$

Therefore, the total resistance is the reciprocal of this expression.

$$R_c = \frac{R_1R_2}{R_1+R_2} \text{ Answer}$$

Number Problems

These problems express the relationship between two numbers.

Example 11

The sum of a number and its reciprocal is $\frac{53}{14}$*. Find the numbers.*

Solution

1. Define variables.

Let $x = a$ number

Then, $\frac{1}{x}$ is the reciprocal of the number

2. Set up an equation.

The equation that describes the relationship between the numbers is: $x + \frac{1}{x} = \frac{53}{14}$

3. Solve the equation.

Find the lowest common denominator.

$$\text{LCM} = 14x$$

Multiply all terms by $14x$

$$14x \cdot x + 14x \cdot \frac{1}{x} = 14x \cdot \frac{53}{14}$$

Cancel common factors in each term.

$$14x \cdot x + 14\cancel{x} \cdot \frac{1}{\cancel{x}} = \cancel{14}x \cdot \frac{53}{\cancel{14}}$$

Simplify.

$$14x^2 + 14 = 53x$$

Write all terms on one side of the equation.

$$14x^2 - 53x + 14 = 0$$

Factor.

$$(7x-2)(2x-7) = 0$$
$$x = \frac{2}{7} \text{ and } x = \frac{7}{2}$$

Notice there are two answers for x, but they are really the same. One answer represents the number and the other answer represents its reciprocal.

4. Check. $\frac{2}{7}+\frac{7}{2}=\frac{4+49}{14}=\frac{53}{14}$. **The answer checks out.**

Work Problems

These are problems where two people or two machines work together to complete a job. Work problems often contain rational expressions. Typically we set up such problems by looking at the part of the task completed by each person or machine. The completed task is the sum of the parts of the tasks completed by each individual or each machine.

Part of task completed by first person + Part of task completed by second person = One completed task

To determine the part of the task completed by each person or machine we use the following fact.

Part of the task completed = rate of work time spent on the task

In general, it is very useful to set up a table where we can list all the known and unknown variables for each person or machine and then combine the parts of the task completed by each person or machine at the end.

Example 12

Mary can paint a house by herself in 12 hours. John can paint a house by himself in 16 hours. How long would it take them to paint the house if they worked together?

Solution:

1. Define variables.

Let $t\ =$ the time it takes Mary and John to paint the house together.

2. Construct a table.

Since Mary takes 12 hours to paint the house by herself, in one hour she paints $\frac{1}{12}$ of the house.

Since John takes 16 hours to pain the house by himself, in one hour he paints $\frac{1}{16}$ of the house.

Mary and John work together for t hours to paint the house together. Using,

Part of the task completed = rate of work • time spent on the task

We can write that Mary completed $\frac{t}{12}$ of the house and John completed $\frac{t}{16}$ of the house in this time.

This information is nicely summarized in the table below:

TABLE 4.7:

Painter	Rate of work (per hour)	Time worked	Part of Task
Mary	$\frac{1}{12}$	t	$\frac{t}{12}$
John	$\frac{1}{16}$	t	$\frac{1}{16}$

3. Set up an equation.

Since Mary completed $\frac{t}{12}$ of the house and John completed $\frac{t}{16}$ and together they paint the whole house in t hours, we can write the equation.

$$\frac{t}{12}+\frac{t}{16}=1.$$

4. Solve the equation.

Find the lowest common denominator.

$$\text{LCM}=48$$

Multiply all terms in the equation by the LCM.

$$48 \cdot \frac{t}{12} + 48 \cdot \frac{t}{16} = 48 \cdot 1$$

Cancel common factors in each term.

$$\cancel{48}^4 . \frac{t}{\cancel{12}} + \cancel{48}.^3 . \frac{t}{\cancel{16}} = 48.1$$

Simplify.

$$4t + 3t = 48$$

$$7t = 48 \Rightarrow t = \frac{48}{7} = 6.86 \textit{ hours} \text{ Answer}$$

Check

The answer is reasonable. We expect the job to take more than half the time Mary takes but less than half the time John takes since Mary works faster than John.

Example 12

Suzie and Mike take two hours to mow a lawn when they work together. It takes Suzie 3.5 hours to mow the same lawn if she works by herself. How long would it take Mike to mow the same lawn if he worked alone?

Solution

1. Define variables.

Let t = the time it takes Mike to mow the lawn by himself.

2. Construct a table.

TABLE 4.8:

Painter	Rate of Work (per Hour)	Time Worked	Part of Task
Suzie	$\frac{1}{3.5} = \frac{2}{7}$	2	$\frac{4}{7}$
Mike	$\frac{1}{t}$	2	$\frac{2}{t}$

3. Set up an equation.

Since Suzie completed $\frac{4}{7}$ of the lawn and Mike completed $\frac{2}{t}$ of the lawn and together they mow the lawn in 2 hours, we can write the equation: $\frac{4}{7} + \frac{2}{t} = 1$.

4. Solve the equation.

Find the lowest common denominator.

$$\text{LCM} = 7t$$

Multiply all terms in the equation by the LCM.

$$7t \cdot \frac{4}{7} + 7t \cdot \frac{2}{t} = 7t \cdot 1$$

Cancel common factors in each term.

$$\cancel{7}t.\frac{4}{\cancel{7}} + \cancel{7}t.\frac{2}{\cancel{t}} = 7t.1$$

Simplify.

$$4t + 14 = 7t$$
$$3t = 14 \Rightarrow t = \frac{14}{3} = 4\frac{2}{3} \textit{ hours} \text{ Answer}$$

Check.

The answer is reasonable. We expect Mike to work slower than Suzie because working by herself it takes her less than twice the time it takes them to work together.

Review Questions

Perform the indicated operation and simplify. Leave the denominator in factored form.

1. $\frac{5}{24} - \frac{7}{24}$
2. $\frac{10}{21} + \frac{9}{35}$
3. $\frac{5}{2x+3} + \frac{3}{2x+3}$
4. $\frac{3x-1}{x+9} - \frac{4x+3}{x+9}$
5. $\frac{4x+7}{2x^2} - \frac{3x-4}{2x^2}$
6. $\frac{x^2}{x+5} - \frac{25}{x+5}$
7. $\frac{2x}{x-4} + \frac{x}{4-x}$
8. $\frac{10}{3x-1} - \frac{7}{1-3x}$
9. $\frac{5}{2x+3} - 3$
10. $\frac{5x+1}{x+4} + 2$
11. $\frac{1}{x} + \frac{2}{3x}$
12. $\frac{4}{5x^2} - \frac{2}{7x^3}$
13. $\frac{4x}{x+1} - \frac{2}{2(x+1)}$
14. $\frac{10}{x+5} + \frac{2}{x+2}$
15. $\frac{2x}{x-3} - \frac{3x}{x+4}$
16. $\frac{4x-3}{2x+1} + \frac{x+2}{x-9}$
17. $\frac{x^2}{x+4} - \frac{3x^2}{4x-1}$
18. $\frac{2}{5x+2} - \frac{x+1}{x^2}$
19. $\frac{x+4}{2x} + \frac{2}{9x}$
20. $\frac{5x+3}{x^2+x} + \frac{2x+1}{x}$
21. $\frac{4}{(x+1)(x-1)} - \frac{5}{(x+1)(x+2)}$
22. $\frac{2x}{(x+2)(3x-4)} + \frac{7x}{(3x-4)^2}$
23. $\frac{3x+5}{x(x-1)} - \frac{9x-1}{(x-1)^2}$

24. $\frac{1}{(x-2)(x-3)}+\frac{4}{(2x+5)(x-6)}$
25. $\frac{3x-2}{x-2}+\frac{1}{x^2-4x+4}$
26. $\frac{-x^2}{x^2-7x+6}+x-4$
27. $\frac{2x}{x^2+10x+25}-\frac{3x}{2x^2+7x-15}$
28. $\frac{1}{x^2-9}+\frac{2}{x^2+5x+6}$
29. $\frac{-x+4}{2x^2-x-15}+\frac{x}{4x^2+8x-5}$
30. $\frac{4}{9x^2-49}-\frac{1}{3x^2+5x-28}$
31. One number is 5 less than another. The sum of their reciprocals is $\frac{13}{36}$. Find the two numbers.
32. One number is 8 times more than another. The difference in their reciprocals is $\frac{21}{20}$. Find the two numbers.
33. A pipe can fill a tank full of oil in 4 hours and another pipe can empty the tank in 8 hours. If the valves to both pipes are open, how long would it take to fill the tank?
34. Stefan could wash the cars by himself in 6 hours and Misha could wash the cars by himself in 5 hours. Stefan starts washing the cars by himself, but he needs to go to his football game after 2.5 hours. Misha continues the task. How long did it take Misha to finish washing the cars?
35. Amanda and her sister Chyna are shoveling snow to clear their driveway. Amanda can clear the snow by herself in three hours and Chyna can clear the snow by herself in four hours. After Amanda has been working by herself for one hour, Chyna joins her and they finish the job together. How long does it take to clear the snow from the driveway?
36. At a soda bottling plant one bottling machine can fulfill the daily quota in 10 hours, and a second machine can fill the daily quota in 14 hours. The two machines start working together but after four hours the slower machine broke and the faster machine had to complete the job by itself. How many hours does the fast machine works by itself?

Review Answers

1. $-\frac{1}{12}$
2. $\frac{11}{15}$
3. $\frac{8}{2x+3}$
4. $\frac{-x-4}{x+9}$
5. $\frac{x=11}{2x^2}$
6. $x-5$
7. $\frac{x}{x-4}$
8. $\frac{17}{3x-1}$
9. $\frac{-6x-4}{2x+3}$
10. $\frac{7x+9}{x+4}$
11. $\frac{5}{3x}$
12. $\frac{28x-10}{35x^3}$
13. $\frac{4x-1}{x+1}$
14. $\frac{12x+30}{(x+5)(x+2)}$
15. $\frac{-x^2+17x}{(x-3)(x+4)}$
16. $\frac{6x^2-34x+19}{(2x+1)(x-9)}$
17. $\frac{x^3-13x^2}{(x+4)(4x-1)}$
18. $-\frac{3x^2+7x+2}{x^2(5x+2)}$
19. $\frac{9x+40}{18x}$
20. $\frac{2x^2+8x+4}{x(x+1)}$
21. $\frac{-x+13}{(x+1)(x-1)(x+2)}$

22. $\frac{13x^2+6x}{(x+2)(3x-4)^2}$
23. $\frac{-6x^2+3x-5}{x(x-1)^2}$
24. $\frac{6x^2-17x-6}{(x-2)(x-3)(2x+5)(x-6)}$
25. $\frac{3x^2-8x+5}{(x-2)^2}$
26. $\frac{-11x^2=34x-24}{(x-6)(x-1)}$
27. $\frac{x^2-21x}{(2x-3)(x+5)^2}$
28. $\frac{3x-4}{(x-3)(x+3)(x+2)}$
29. $\frac{-x^2+4x=4}{(2x+5)(x-3)(2x-1)}$
30. $\frac{x+9}{(3x+7)(3x-7)(x+4)}$
31. $x = 4, x+5 = 4$ or $x = -\frac{45}{13}, x+5 = \frac{20}{13}$
32. $x = \frac{5}{6}, 8x = \frac{20}{3}$
33. 8 hours
34. 2 hours and 55 minutes
35. $1\frac{1}{7}$ *hours*, or 1 hour 9 minutes
36. $3\frac{1}{7}$ *hours*

4.7 Solutions of Rational Equations

Learning Objectives

- Solve rational equations using cross products.
- Solve rational equations using lowest common denominators.
- Solve real-world problems with rational equations.

Introduction

A **rational equation** is one that contains rational expressions. It can be an equation that contains rational coefficients or an equation that contains rational terms where the variable appears in the denominator.

An example of the first kind of equation is $\frac{3}{5}x+\frac{1}{2}=4$.

An example of the second kind of equation is $\frac{x}{x-1}+1=\frac{4}{2x+3}$.

The first aim in solving a rational equation is to eliminate all denominators. In this way, we can change a rational equation to a polynomial equation which we can solve with the methods we have learned this far.

Solve Rational Equations Using Cross Products

A rational equation that contains two terms is easily solved by the method of **cross products** or **cross multiplication**. Consider the following equation.

$$\frac{x}{5}=\frac{x+1}{2}$$

Our first goal is to eliminate the denominators of both rational expressions. In order to remove the five from the denominator of the first fraction, we multiply both sides of the equation by five:

$$\cancel{5}\cdot\frac{x}{\cancel{5}}=\frac{x+1}{2}\cdot 5$$

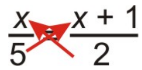
Now, we remove the 2 from the denominator of the second fraction by multiplying both sides of the equation by two.

$$2\cdot x=\frac{5(x+1)}{\cancel{2}}\cancel{2}$$

The equation simplifies to $2x=5(x+1)$.

$$2x=5x+\Rightarrow x=-\frac{5}{3}\text{ Answer}$$

Notice that when we remove the denominators from the rational expressions we end up multiplying the numerator on one side of the equal sign with the denominator of the opposite fraction.



Once again, we obtain the simplified equation: $2x = 5(x+1)$, whose solution is $x = -\frac{5}{3}$

We check the answer by plugging the answer back into the original equation.

Check

On the left-hand side, if $x = -\frac{5}{3}$, then we have

$$\frac{x}{5} = \frac{\frac{-5}{3}}{5} = -\frac{1}{3}$$

On the right hand side, we have

$$\frac{x+1}{2} = \frac{\frac{-5}{3+1}}{2} = \frac{\frac{-2}{3}}{2} = -\frac{1}{3}$$

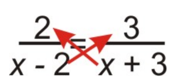
Since the two expressions are equal, the answer checks out.

Example 1

Solve the equation $\frac{2}{x-2} = \frac{3}{x+3}$.

Solution

Use cross-multiplication to eliminate the denominators of both fractions.



The equation simplifies to

$$2(x+3) = 3(x-2)$$

Simplify.

$$\begin{aligned} 2x+6 &= 3x-6 \\ x &= 12 \end{aligned}$$

Check.

$$\begin{aligned} \frac{2}{x-2} &= \frac{2}{12-2} = \frac{2}{10} = \frac{1}{5} \\ \frac{3}{x+3} &= \frac{3}{12+3} = \frac{3}{15} = \frac{1}{5} \end{aligned}$$

The answer checks out.

Example 2

Solve the equation $\frac{23}{x+4} = \frac{5}{x}$.

Solution

Cross-multiply.

The equation simplifies to

$$2x^2 = 5(x+4)$$

Simplify.

$$2x^2 = 5x+20$$

Move all terms to one side of the equation.

$$2x^2 - 5x - 20 = 0$$

Notice that this equation has a degree of two, that is, it is a *quadratic equation*. We can solve it using the quadratic formula.

$x = \frac{5 \pm \sqrt{185}}{4} \Rightarrow x \approx -2.15$ or $x \approx 4.65$

Answer

It is important to check the answer in the original equation when the variable appears in any denominator of the equation because the answer might be an excluded value of any of the rational expression. If the answer obtained makes any denominator equal to zero, that value is not a solution to the equation.

Check:

First we check $x = -2.15$ by substituting it in the original equations. On the left hand side we get the following.

$$\frac{2x}{x+4} = \frac{2(-2.15)}{-2.15+4} \frac{-4.30}{1.85} = -2.3$$

Now, check on the right hand side.

$$\frac{5}{x} = \frac{5}{-2.15} = -2.3$$

Thus, 2.15 checks out.

For $x = 4.65$ we repeat the procedure.

$$\frac{2x}{x+4} = \frac{2(4.65)}{4.65+4} = 1.08.$$
$$\frac{5}{x} = \frac{5}{4.65} = 1.08.$$

4.65 also checks out.

Solve Rational Equations Using the Lowest Common Denominators

An alternate way of eliminating the denominators in a rational equation is to multiply all terms in the equation by the lowest common denominator. This method is suitable even when there are more than two terms in the equation.

Example 3

Solve $\frac{3x}{35} = \frac{x^2}{5} - \frac{1}{21}$.

Solution

Find the lowest common denominator:

LCM = 105

Multiply each term by the LCD.

$$105 \cdot \frac{3x}{35} = 105 \cdot \frac{x^2}{5} - 105 \cdot \frac{1}{21}$$

Cancel common factors.

$$\cancel{105}^{3} . \frac{3x}{\cancel{35}} = \cancel{105}^{21} . \frac{x^2}{\cancel{5}}{}^{5} - \cancel{105} . \frac{1}{\cancel{21}}$$

The equation simplifies to

$$9x = 21x^2 - 5$$

Move all terms to one side of the equation.

$$21x^2 - 9x - 5 = 0$$

Solve using the quadratic formula.

$$x = \frac{9 \pm \sqrt{501}}{42}$$

$x \approx -0.32$ or $x \approx 0.75$ Answer

Check

We use the substitution $x = -0.32$.

$$\frac{3x}{35} = \frac{3(-0.32)}{35} = -0.27$$
$$\frac{x^2}{5} - \frac{1}{24} = \frac{(-0.32)^2}{5} - \frac{1}{21} = -.027. \text{ The answer checks out.}$$

Now we check the solution $x = 0.75$.

$$\frac{3x}{35} = \frac{3(0.75)}{35} = 0.64$$
$$\frac{x^2}{5} - \frac{1}{21}\frac{(0.75)^2}{5} - \frac{1}{21} = .064. \text{ The answer checks out.}$$

Example 4

Solve $\frac{3}{x+2} - \frac{4}{x=5} = \frac{2}{x^2-3x-10}$.

Solution

Factor all denominators.

$$\frac{3}{x+2} - \frac{4}{x-5} - \frac{2}{(x+2)(x-5)}$$

Find the lowest common denominator.

$$\text{LCM} = (x+2)(x-5)$$

Multiply all terms in the equation by the LCM.

$$(x+2)(x-5)\cdot\frac{3}{x+2} - (x+2)(x-5)\cdot\frac{4}{x-5} = (x+2)(x-5)\cdot\frac{2}{(x+2)(x-5)}$$

Cancel the common terms.

$$\cancel{(x+2)}(x-5)\cdot\frac{3}{\cancel{x+2}} - (x+2)\cancel{(x-5)}\cdot\frac{4}{\cancel{x-5}} = \cancel{(x+2)(x-5)}\cdot\frac{2}{\cancel{(x+2)(x-5)}}$$

The equation simplifies to

$$3(x-5) - 4(x+2) = 2$$

Simplify.

$$3x - 15 - 4x - 8 = 2$$
$$x = -25 \text{ Answer}$$

Check.

$$\frac{3}{x+2}-\frac{4}{x-5}=\frac{3}{-25+2}-\frac{4}{-25-5}=0.003$$
$$\frac{2}{x^2-3x-10}=\frac{2}{(-25)^2-3(-25)-10}=0.003$$

The answer checks out.

Example 5

Solve $\frac{2x}{2x+1}+\frac{x}{x+4}=1$.

Solution

Find the lowest common denominator.

$$\text{LCM}=(2x+1)(x+4)$$

Multiply all terms in the equation by the LCM.

$$(2x+1)(x+4)\cdot\frac{2x}{2x+1}+(2x+1)(x+4)\cdot\frac{x}{x+4}=(2x+1)(x+4)$$

Cancel all common terms.

$$\cancel{(2x+1)}(x+4)\cdot\frac{2x}{\cancel{2x+1}}+(2x+1)\cancel{(x+4)}\cdot\frac{x}{\cancel{x+4}}=(2x+1)(x+4)$$

The simplified equation is

$$2x(x+4)+x(2x+1)=(2x+1)(x+4)$$

Eliminate parentheses.

$$2x^2+8x+2x^2+x=2x^2+9x+4$$

Collect like terms.

$$2x^2=4$$
$$x^2=2\Rightarrow x=\pm\sqrt{2}\text{ Answer}$$

Check.

$$\frac{2x}{2x+1}+\frac{x}{x+4}=\frac{2\sqrt{2}}{2\sqrt{2}+1}+\frac{\sqrt{2}}{\sqrt{2}+4}\approx 0.739+0.261=1.\text{ The answer checks out.}$$
$$\frac{2x}{2x+1}+\frac{x}{x+4}=\frac{2\left(-\sqrt{2}\right)}{2\left(-\sqrt{2}\right)+1}+\frac{-\sqrt{2}}{-\sqrt{2}+4}\approx 1.547+0.547=1.\text{ The answer checks out.}$$

Solve Real-World Problems Using Rational Equations

Motion Problems

A motion problem with no acceleration is described by the formula distance $=$ speed $\times$ time.

These problems can involve the addition and subtraction of rational expressions.

Example 6

Last weekend Nadia went canoeing on the Snake River. The current of the river is three miles per hour. It took Nadia the same amount of time to travel 12 miles downstream as three miles upstream. Determine the speed at which Nadias canoe would travel in still water.

Solution

1. Define variables

Let $s =$ speed of the canoe in still water

Then, $s+3$ $=$ the speed of the canoe traveling downstream

$s-3$ $=$ the speed of the canoe traveling upstream

2. Construct a table.

We make a table that displays the information we have in a clear manner:

TABLE 4.9:

Direction	Distance (miles)	Rate	Time
Downstream	12	$s+3$	t
Upstream	3	$s-3$	t

3. Write an equation.

Since distance $=$ rate $\times$ time, we can say that: time $= \frac{\text{distance}}{\text{rate}}$.

The time to go downstream is

$$t = \frac{12}{s+3}$$

The time to go upstream is

$$t = \frac{3}{s-3}$$

Since the time it takes to go upstream and downstream are the same then: $\frac{3}{s-3} = \frac{12}{s+3}$

4. Solve the equation

Cross-multiply.

$$3(s+3) = 12(s-3)$$

Simplify.

$$3s+9=12s-36$$

Solve.

$$s=5\ mi/hr \text{ Answer}$$

5. Check

Upstream: $t=\frac{12}{8}=1\frac{1}{2}\ hour$; Downstream: $t=\frac{3}{2}=1\frac{1}{2}\ hour$. **The answer checks out.**

Example 8

Peter rides his bicycle. When he pedals uphill he averages a speed of eight miles per hour, when he pedals downhill he averages 14 miles per hour. If the total distance he travels is 40 miles and the total time he rides is four hours, how long did he ride at each speed?

Solution

1. Define variables.

Let t_1 = time Peter bikes uphill, t_2 = time Peter bikes downhill, and d = distance he rides uphill.

2. Construct a table

We make a table that displays the information we have in a clear manner:

TABLE 4.10:

Direction	Distance (miles)	Rate (mph)	Time (hours)
Uphill	d	8	t_1
Downhill	$40-d$	14	t_2

3. Write an equation

We know that

$$\text{time}=\frac{\text{distance}}{\text{rate}}$$

The time to go uphill is

$$t_1=\frac{d}{8}$$

The time to go downhill is

$$t_2=\frac{40-d}{14}$$

We also know that the total time is 4 hours.

$$\frac{d}{8}+\frac{40-d}{14}=4$$

4. Solve the equation.

Find the lowest common denominator:

LCM = 56

Multiply all terms by the common denominator:

$$\begin{aligned} 56\cdot\frac{d}{8}+56\cdot\frac{40-d}{14}&=4\cdot 56\\ 7d+160-4d&=224\\ 3d&=64 \end{aligned}$$

Solve.

$$d=21.3\ miles \text{ Answer}$$

5. Check.

Uphill: $t=\frac{21.3}{8}=2.67\ hours$; Downhill: $t=\frac{40-21.3}{14}=1.33\ hours$. **The answer checks out.**

Shares

Example 8

A group of friends decided to pool together and buy a birthday gift that cost $200. Later 12 of the friends decided not to participate any more. This meant that each person paid $15 more than the original share. How many people were in the group to start?

Solution

1. Define variables.

Let x = the number of friends in the original group

2. Make a table.

We make a table that displays the information we have in a clear manner:

TABLE 4.11:

	Number of People	Gift Price	Share Amount
Original group	x	200	$\frac{200}{x}$
Later group	$x-12$	200	$\frac{200}{x}$

3. Write an equation.

Since each persons share went up by $15 after 2 people refused to pay, we write the equation:

$$\frac{200}{x-12}=\frac{200}{x}+15$$

4. Solve the equation.

Find the lowest common denominator.

$$\text{LCM} = x(x-12)$$

Multiply all terms by the LCM.

$$x(x-12)\cdot\frac{200}{x-12} = x(x-12)\cdot\frac{200}{x} + x(x-12)\cdot 15$$

Cancel common factors in each term:

$$\cancel{x(x-12)}\cdot\frac{200}{\cancel{x-12}} = \cancel{x}(x-12)\cdot\frac{200}{\cancel{x}} + x(x-12)\cdot 15$$

Simplify.

$$200x = 200(x-12) + 15x(x-12)$$

Eliminate parentheses.

$$200x = 200x - 2400 + 15x^2 - 180x$$

Collect all terms on one side of the equation.

$$0 = 15x^2 - 180x - 2400$$

Divide all terms by 15.

$$0 = x^2 - 12x - 160$$

Factor.

$$0 = (x-20)(x+8)$$

Solve.

$$x = 20, x = -8$$

The answer is $x = 20$ people. We discard the negative solution since it does not make sense in the context of this problem.

5. Check.

Originally \$200 shared among 20 people is \$10 each. After 12 people leave, \$200 shared among 8 people is \$25 each. So each person pays \$15 more.

The answer checks out.

Review Questions

Solve the following equations.

Solve the following equations.

$\frac{2x+1}{4} = \frac{x-3}{10}$

$\frac{4x}{x+2} = \frac{5}{9}$

$\frac{5}{3x-4} = \frac{2}{x+1}$

$\frac{7x}{x-5} = \frac{x+3}{x}$

$\frac{2}{x+3} - \frac{1}{x+4} = 0$

$\frac{3x^2+2x-1}{x^2-1} = -2$

$x + \frac{1}{x} = 2$

$-3 + \frac{1}{x+1} = \frac{2}{x}$

$\frac{1}{x} - \frac{x}{x-2} = 2$

$\frac{3}{2x-1} + \frac{2}{x+4} = 2$

$\frac{2x}{x-1} - \frac{x}{3x+4} = 3$

$\frac{x+1}{x-1} + \frac{x-4}{x+4} = 3$

$\frac{x}{x-2} + \frac{x}{x+3} = \frac{1}{x^2+x-6}$

$\frac{2}{x^2+4x+3} = 2 + \frac{x-2}{x+3}$

$\frac{1}{x+5} - \frac{1}{x-5} = \frac{1-x}{x+5}$

$\frac{x}{x^2-36} + \frac{1}{x-6} = \frac{1}{x+6}$

$\frac{2x}{3x+3} - \frac{1}{4x+4} = \frac{2}{x+1}$

$\frac{-x}{x-2} + \frac{3x-1}{x+4} = \frac{1}{x^2+2x-8}$

Juan jogs a certain distance and then walks a certain distance. When he jogs he averages 7 miles/hour. When he walks, he averages 3.5 miles/hour. If he walks and jogs a total of 6 miles in a total of 7 hours, how far does he jog and how far does he walk?

A boat travels 60 miles downstream in the same time as it takes it to travel 40 miles upstream. The boat's speed in still water is 20 miles/hour. Find the speed of the current.

Paul leaves San Diego driving at 50 miles/hour. Two hours later, his mother realizes that he forgot something and drives in the same direction at 70 miles/hour. How long does it take her to catch up to Paul?

On a trip, an airplane flies at a steady speed against the wind. On the return trip the airplane flies with the wind. The airplane takes the same amount of time to fly 300 miles against the wind as it takes to fly 420 miles with the wind. The wind is blowing at 30 miles/hour. What is the speed of the airplane when there is no wind?

A debt of $420 is shared equally by a group of friends. When five of the friends decide not to pay, the share of the other friends goes up by $25. How many friends were in the group originally?

A non-profit organization collected $2250 in equal donations from their members to share the cost of improving a park. If there were thirty more members, then each member could contribute $20 less. How many members does this organization have?

Review Answers

1. $x = -\frac{11}{8}$
2. $x = \frac{10}{31}$
3. $x = 13$
4. no real solution
5. $x = -5$
6. $x = \frac{3}{5}$
7. $x = 1$
8. $x = \frac{1}{3}$
9. $x = 1, x = \frac{2}{3}$
10. $x = -3.17, x = 1.42$
11. $x = -1.14, x = 2.64$
12. $x = -10.84, x = 1.84$
13. $x = -1, x = \frac{1}{2}$
14. $x = -2, x = -\frac{1}{3}$
15. $x = -.74, x = 6.74$
16. $x = -12$
17. $x = \frac{27}{8}$
18. $x = .092, x = 5.41$
19. jogs 3.6 miles and walks 2.4 miles
20. 4 miles/hour
21. 5 hours
22. 180 miles/hour
23. 12 friends
24. 45 members

4.8 References

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Chapter 5 Exponential and Logarithmic Equations and Functions

Chapter Outline

5.1 Composite Functions and Inverse Functions

Learning objectives

- Evaluate and find composite functions.
- Find the inverse of a function
- Determine if a function is invertible
- State the domain and range for a function and its inverse
- Graph functions and their inverses
- Use composition to verify if two functions are inverses.

Introduction

In this chapter, we will focus on two related functions: exponential functions,and logarithmic functions. These two functions have a special relationship with one another: they are **inverses** of each other. In this first lesson we will develop the idea of inverses, both algebraically and graphically, as background for studying these two types of functions in depth. We will begin with a familiar, every-day example of two functions that are inverses.

Composite Functions

A composite function or composition of functions is a function made up of more than one function. A composite function can be thought of as a function inside another function. The notation used for composition of functions is:

$(f \circ g)(x) = f(g(x))$

To calculate a composite function, we evaluate the inner function and then substitute the result into the outer function. We can say the output of the inner function becomes the input into the outer function.

Sometimes we find composite function values if the intial intial input is a given value or real number. In this case the composite function value is a real number. However, if the intial input is a variable or variable expression, the composite function is another function. Let's look at examples of both cases.

Example 1: Find the composite function values $(f \circ g)(3)$ and $(g \circ f)(3)$, given $f(x) = x^2 - 2x + 1$ and $g(x) = x - 5$.

a) $(f \circ g)(3) = f(g(3))$

We start by determining the inner function value, $g(3)$.

$$g(3) = 3 - 5 = -2$$

Now substitute -2 for $g(3)$ in the composite function.

$$f(g(3)) = f(-2)$$

Now we find the function value $f(-2)$.

$$f(-2) = (-2)^2 - 2(-2) + 1 = 4 + 4 + 1 = 9$$

Therefore, $f(g(3)) = 9$

b) $(g \circ f)(3) = g(f(3))$

We start by determining the inner function value, $f(3)$.

$$f(3) = (3)^2 - 2(3) + 1 = 9 - 6 + 1 = 4$$

Now substitute 4 for $f(3)$ in the composite function.

$$g(f(3)) = g(4)$$

Now we find the function value $g(4)$.

$$g(4) = 4 - 5 = -1$$

Therefore, $g(f(3)) = -1$

Notice the result is a real number value.

Now we will look at a composition of functions that results in another function.

Example 2: Find the composite functions $(f \circ g)(x)$ and $(g \circ f)(x)$, given $f(x) = x^2 - 2x + 1$ and $g(x) = x - 5$.

a) $(f \circ g)(x) = f(g(x))$

Notice we can not evaluate $g(x)$ because we are not given an input for $g(x)$. As a result, the input into $f(x)$ is $g(x)$.

$$f(g(x)) = f(x-5) = (x-5)^2 - 2(x-5) + 1 = x^2 - 10x + 25 - 2x + 10 + 1 = x^2 - 12x + 36$$

$$f(g(x)) = x^2 - 12x + 36$$

Notice the composite function is another function.

b) $(g \circ f)(x) = g(f(x))$

Notice we can not evaluate $f(x)$ because we are not given an input for $f(x)$. As a result, the input into $g(x)$ is $f(x)$.

$$g(f(x)) = g(x^2 - 2x + 1) = x^2 - 2x + 1 - 5 = x^2 - 2x - 4$$

$$g(f(x)) = x^2 - 2x - 4$$

Notice the composite function is another function.

Functions and inverses

In the United States, we measure temperature using the Fahrenheit scale. In other countries, people use the Celsius scale. The equation C = 5/9 (F - 32) can be used to find C, the Celsius temperature, given F, the Fahrenheit temperature. If we write this equation using function notation, we have $t(x) = \frac{5}{9}(x-32)$. The input of the function is a Fahrenheit temperature, and the output is a Celsius temperature. For example, the freezing point on the Fahrenheit scale is 32 degrees. We can find the corresponding Celsius temperature using the function:

$$t(32) = \frac{5}{9}(32-32) = \frac{5}{9}\cdot 0 = 0$$

This function allows us to convert a Fahrenheit temperature into Celsius, but what if we want to convert from Celsius to Fahrenheit?

Consider again the equation above: $C = \frac{5}{9}(F-32)$. We can solve this equation to isolate F:

TABLE 5.1:

$C = \frac{5}{9}(F-32)$ $\frac{9}{5}C = \frac{9}{5}\times\frac{5}{9}(F-32)$ $\frac{9}{5}C = F-32$ $\frac{9}{5}C+32 = F$

If we write this equation using function notation, we get $f(x) = \frac{9}{5}x+32$. For this function, the input is the Celsius temperature, and the output is the Fahrenheit temperature. For example, if $x = 0, f(0) = \frac{9}{5}(0)+32 = 0+32 = 32$.

Now consider the functions $t(x) = \frac{5}{9}(x-32)$ and $f(x) = \frac{9}{5}x+32$ together. The input of one function is the output of the other. This is an informal way of saying that these functions are **inverses.** Formally, the inverse of a function is defined as follows:

Inverse Function

Functions $f(x)$ and $g(x)$ are inverses if

$f(g(x)) = g(f(x)) = x$ which can also be written $f\circ g = g\circ f = x$.

The following notation is used to indicate inverse functions:

If $f(x)$ and $g(x)$ are inverses, then

$f(x) = g^{-1}(x)$ and $g(x) = f^{-1}(x)$ with can also be written $f = g^{-1}$ and $g = f^{-1}$.

Note: $f^{-1}(x)$ does **not** equal $\frac{1}{f(x)}$.

Informally, we define the inverse of a function as the relation we obtain by switching the domain and range of the function. Because of this definition, you can find an inverse by switching the roles of x and y in an equation. For example, consider the function $g(x) = 2x$. This is the line $y = 2x$. If we switch x and y, we get the equation $x = 2y$. Dividing both sides by 2, we get y = 1/2 x. Therefore the functions $g(x) = 2x$ and y = 1/2 x are inverses. Using function notation, we can write y = 1/2 x as $g^{-1}(x)$ = 1/2 x.

Example 3: Find the inverse of each function.

a. $f(x) = 5x-8$

b. $f(x) = x^3$

a. First write the function using $y =$ notation, then interchange x and y:**Solution:**

$f(x) = 5x - 8 \Rightarrow y = 5x - 8$

So the inverse is: $x = 5y - 8$

Then isolate y:

$x = 5y - 8$ (Add 8 to both sides.)

$x + 8 = 5y$ (Divide both sides by 5.)

$y = \frac{1}{5}x + \frac{8}{5}$

$f^{-1}(x) = \frac{1}{5}x + \frac{8}{5}$ (Written using inverse function notation)

b.First write the function using $y =$ notation, the interchagne x and y.

$f(x) = x^3 \Rightarrow y = x^3$

So the inverse is: $x = y^3$

Then isolate y:

$y = \sqrt[3]{x}$ (Cube root both sides.)

$y = \sqrt[3]{x}$

$f^{-1}(x) = \sqrt[3]{x}$ (Written using inverse function notation)

Because of the definition of inverse, the graphs of inverses are reflections across the line $y = x$. The graph below shows $t(x) = \frac{5}{9}(x - 32)$ and $f(x) = \frac{9}{5}x + 32$ on the same graph, along with the reflection line $y = x$.

A note about graphing with software or a graphing calculator: if you look at the graph above, you can see that the lines are reflections over the line $y = x$. However, if you do not view the graph in a window that shows equal scales of the x- and y-axes, the graph might not look like this.

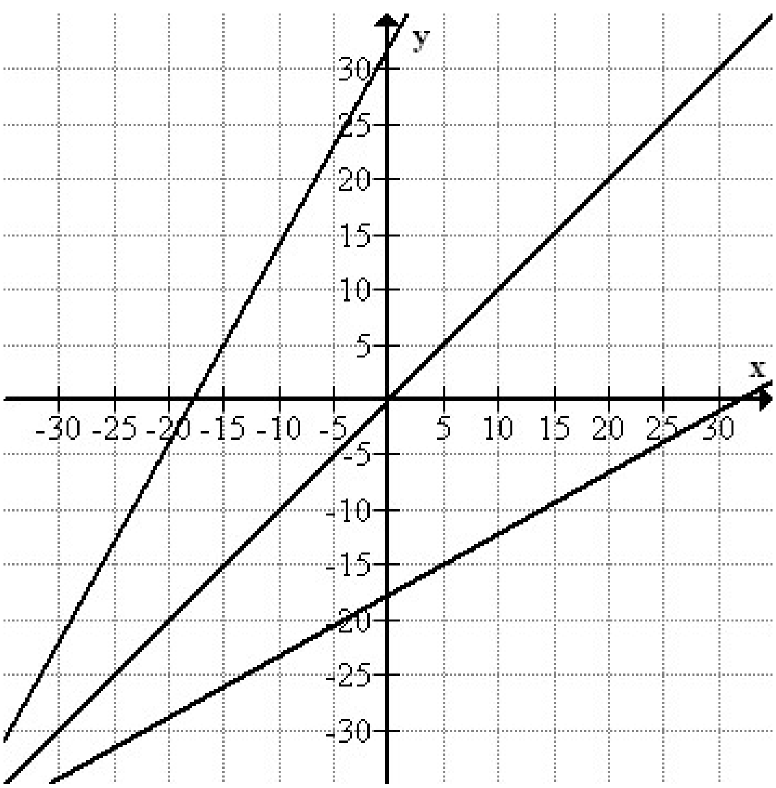


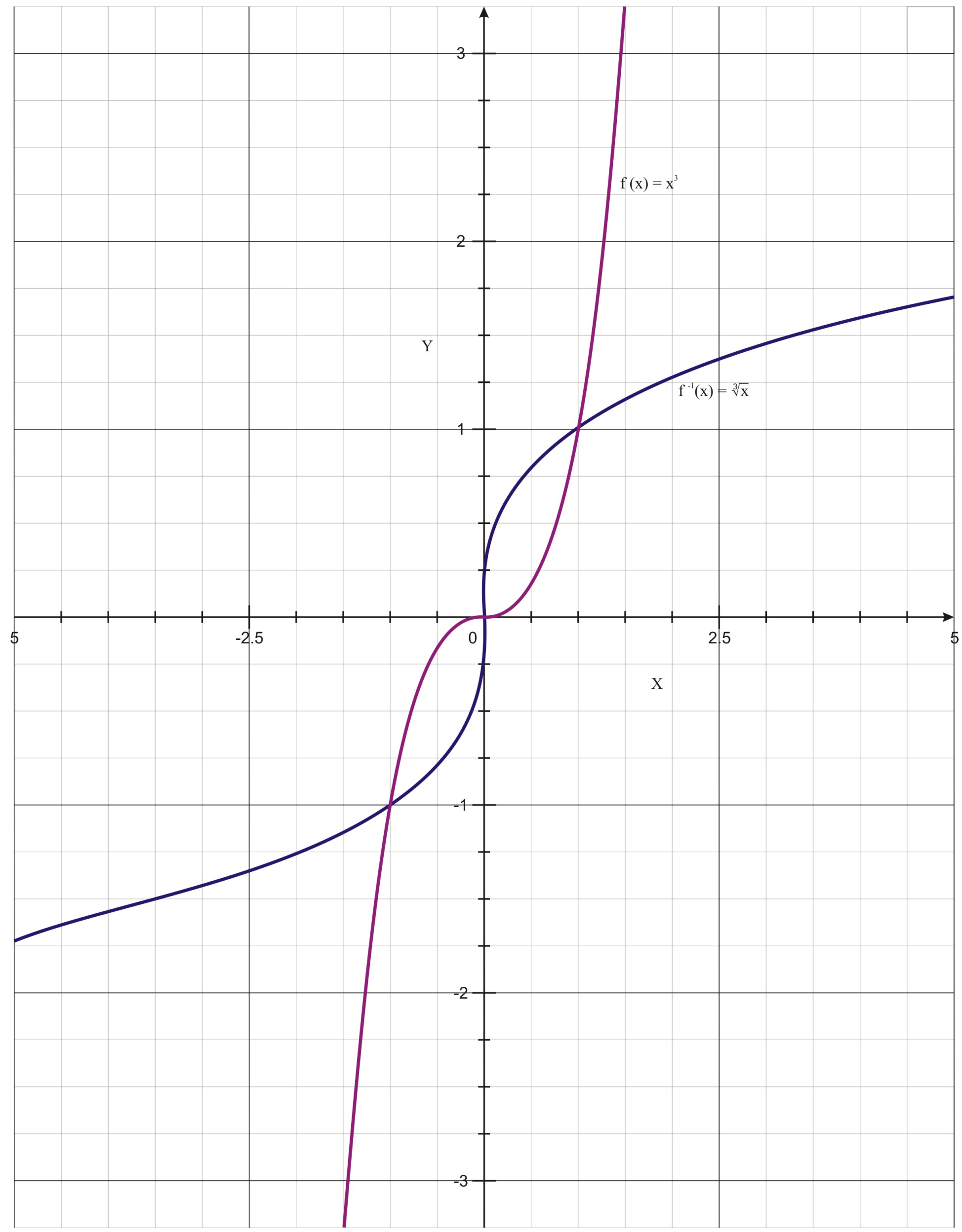
FIGURE 5.1

Before continuing, there are two other important things to note about inverses. First, remember that the ' -1' is not an exponent, but a symbol that represents an inverse. Second, not every function has an inverse that is a function.

In the examples we have considered so far, we inverted a function, and the resulting relation was also a function. However, some functions are not **invertible**; that is, following the process of "inverting" them does not produce a relation that is a function. We will return to this issue below when we examine domain and range of functions and their inverses. First we will look at a set of functions that *are* invertible.

Inverses of 1-to-1 functions

Consider again example 1 above. We began with the function $f(x) = x^3$, and we found the inverse $f^{-1}(x) = \sqrt[3]{x}$. The graphs of these functions are show below.

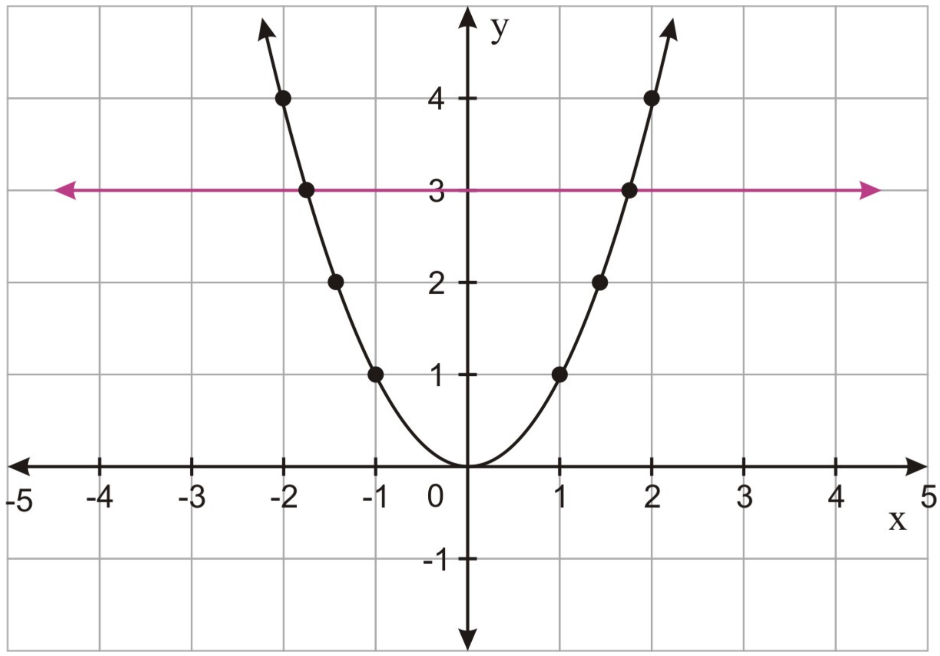


The function $f(x) = x^3$ is an example of a **one-to-one function,** which is defined as follows:

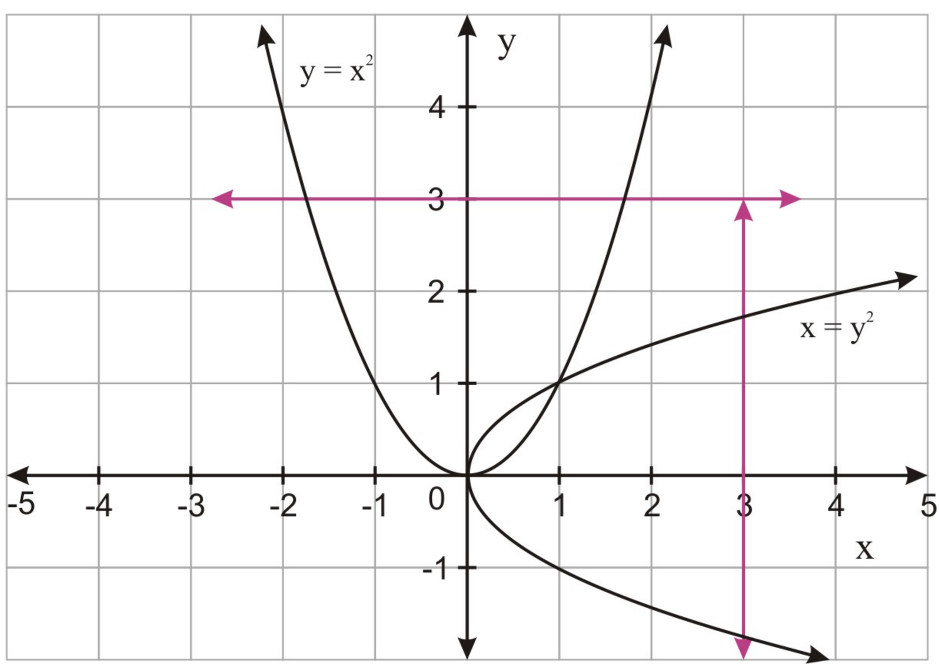
TABLE 5.2:

One to one
A function is **one-to-one** if and only if every element of its domain corresponds to *exactly* one element of its range.

The linear functions we examined above are also one-to-one. The function $y = x^2$, however, is not one-to-one. The graph of this function is shown below.



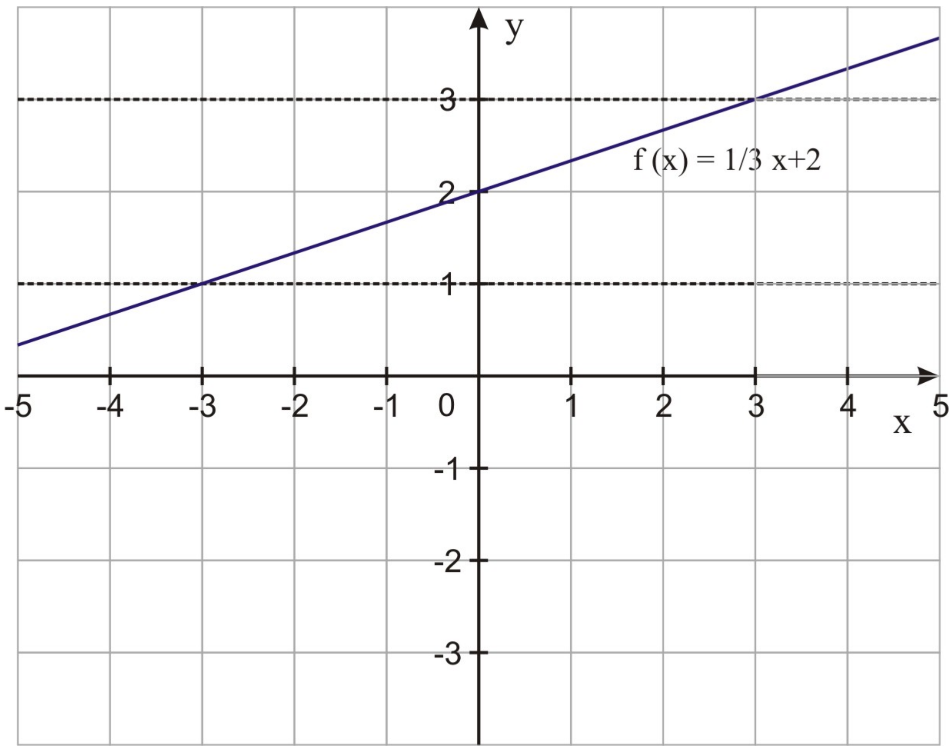
You may recall that you can identify a relation as a function if you draw a vertical line through the graph, and the line touches only one point. Notice then that if we draw a horizontal line through $y = x^2$, the line touches more than one point. Therefore if we inverted the function, the resulting graph would be a reflection over the line $y = x$, and the inverse would not be a function. It fails the vertical line test.



The function $y = x^2$ is therefore *not* a one-to-one function. A function that *is* one-to-one will be invertible. You can determine this graphically by drawing a horizontal line through the graph of the function. For example, if you draw a horizontal line through the graph of $f(x) = x^3$, the line will only touch one point on the graph, no matter where you draw the line.

Example 4: Graph the function $f(x) = \frac{1}{3}x + 2$. Use a horizontal line test to verify that the function is invertible.

Solution: The graph below shows that this function is invertible. We can draw a horizontal line at any y value, and the line will only cross $f(x) = \frac{1}{3}x + 2$ once.



In sum, a one-to-one function is invertible. That is, if we invert a one-to-one function, its inverse is also a function. Now that we have established what it means for a function to be invertible, we will focus on the domain and range of inverse functions.

Domain and range of functions and their inverses

Because of the definition of inverse, a functions domain is its inverses range, and the inverses domain is the functions range. This statement may seem confusing without a specific example.

Example 3: State the domain and range of the function and its inverse:

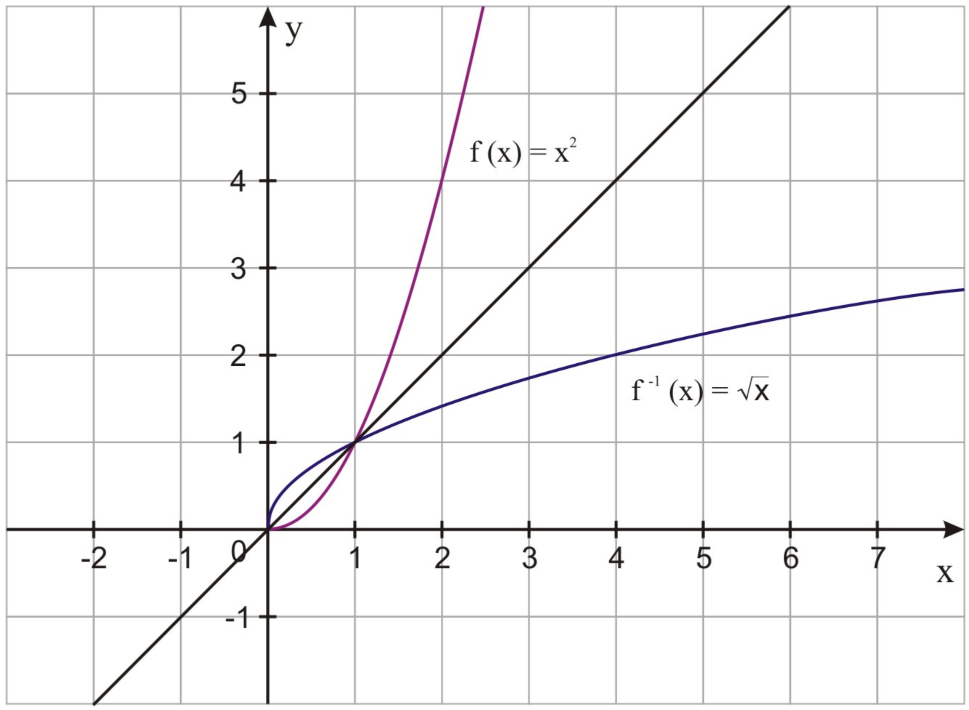
Function: *(1, 2), (2, 5), (3, 7)*

Solution: the inverse of this function is the set of points *(2, 1), (5, 2), (7, 3)*

The domain of the function is *{1, 2, 3}*. This is also the range of the inverse. The range of the function is *{2, 5, 7}*. This is also the domain of the inverse.

The linear functions we examined previously, as well as $f(x) = x^3$, all had domain and range both equal to the set of all real numbers. Therefore the inverses also had domain and range equal to the set of all real numbers. Because the domain and range were the same for these functions, switching them maintained that relationship.

Also, as we found above, the function $y = x^2$ is not one-to-one, and hence it is not invertible. That is, if we invert it, the resulting relation is not a function. We can change this situation if we define the domain of the function in a more limited way. Let $f(x)$ be a function defined as follows: $f(x) = x^2$, with domain limited to real numbers 0. Then the inverse of the function is the square root function: $f^{-1}(x) = \sqrt{x}$



Example 5: Define the domain for the function $f(x) = (x - 2)^2$ so that f is invertible.

Solution: The graph of this function is a parabola. We need to limit the domain to one side of the parabola. Conventionally in cases like these we choose the positive side; therefore, the domain is limited to real numbers 2.

Inverse functions and composition

In the examples we have considered so far, we have taken a function and found its inverse. We can also analyze two functions and determine whether or not they are inverses. Recall the formal definition from above:

Two functions $f(x)$ and $g(x)$ are inverses if and only if $f(g(x)) = g(f(x)) = x$.

This definition is perhaps easier to understand if we look at a specific example. Lets use two functions that we have established as inverses: $f(x) = 2x$ and $g(x) = 1/2\ x$. Lets also consider a specific x value. Let $x = 8$. Then we have $f(g(8)) = f(1/2\ 8) = f(4) = 2(4) = 8$. Similarly we could establish that $g(f(8)) = 8$. Notice that there is nothing special about $x = 8$. For any x value we input into f, the same value will be output by the composed functions:

$$f(g(x)) = f\left(\tfrac{1}{2}x\right) = 2\left(\tfrac{1}{2}x\right) = x$$

$$g(f(x)) = g(2x) = \tfrac{1}{2}(2x) = x$$

Example 6: Use composition of functions to determine if $f(x) = 2x + 3$ and $g(x) = 3x - 2$ are inverses.

Solution: The functions are not inverses.

We only need to check one of the compositions: $f(g(x)) = f(3x-2) = 2(3x-2)+3 = 6x-4+3 = 6x-1 \neq x$

Lesson Summary

In this lesson we have defined the concept of inverse, and we have examined functions and their inverses, both algebraically and graphically. We established that functions that are one-to-one are invertible, while other functions are not necessarily invertible. (However, we can redefine the domain of a function such that it is invertible.) In the remainder of the chapter we will examine two families of functions whose members are inverses.

Points to Consider

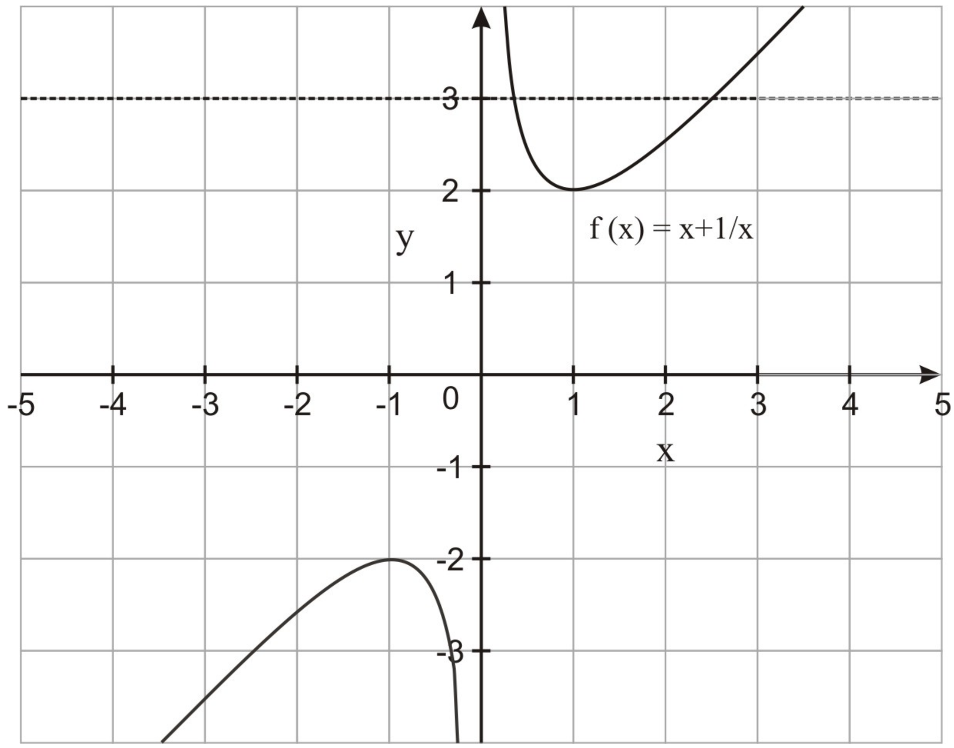
1. Can a function be its own inverse? If so, how?
2. Consider the other function families you learned about in chapter 1. What do their inverses look like?
3. How is the rate of change of a function related to the rate of change of the functions inverse?

Review Questions

1. Find the inverse of the function $f(x) = \frac{1}{2}x - 7$.
2. Use the horizontal line test to determine if the function $f(x) = x + \frac{1}{x}$ is invertible or not.
3. Use composition of functions to determine if the functions are inverses: $g(x) = 2x - 6$ and $h(x) = \frac{1}{2}x + 3$.
4. Use composition of functions to determine if the functions are inverses: $f(x) = x + 2$ and $p(x) = x - \frac{1}{2}$.
5. Given the function $f(x) = (x+1)^2$, how should the domain be restricted so that the function is invertible?
6. Consider the function $f(x) = \frac{3}{2}x + 4$.
   a. Find the inverse of the function.
   b. State the slope of the function and its inverse. What do you notice?
7. Given the function (0, 5), (1, 7), (2, 13), (3, 19)
   a. Find the inverse of the function.
   b. State the domain and range of the function.
   c. State the domain and range of the inverse.
8. Consider the function *a(x)* =
9. Consider the function *f(x)* = *c*, where *c* is a real number. What is the inverse? Is *f* invertible? Explain.
10. A store sells fabric by the length. Red velvet goes on sale after Valentines day for $4.00 per foot.
    a. Write a function to model the cost of *x* feet of red velvet.
    b. What is the inverse of this function?
    c. What does the inverse represent?

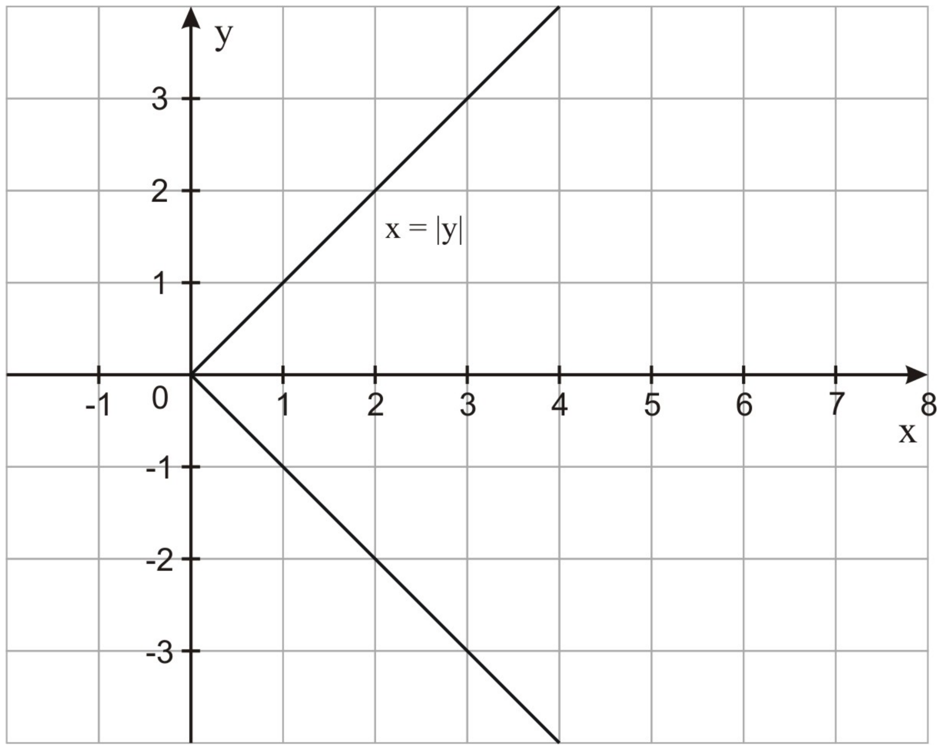
Review Answers

1. $y = 2x + 14$

2. 

The function is not invertible.

3. The functions are inverses.
$g(h(x)) = g\left(\frac{1}{2}x+3\right) = 2\left(\frac{1}{2}x+3\right) - 6 = x+6-6 = x$
$h(g(x)) = h(2x-6) = \frac{1}{2}(2x-6)+3 = x-3+3 = x$
4. The functions are not inverses.
$f(p(x)) = (x-\frac{1}{2})+2 = x+\frac{3}{2} \neq x$
5. $x \geq -1$
6. a. $y = \frac{2}{3}x - \frac{8}{3}$
b. The slope of the function is 3/2 and the slope of the inverse is 2/3. The slopes are reciprocals.
7. a. (5, 0), (7, 1), (13, 2), (19, 3)
b. Domain: {0, 1, 2, 3} and Range: {5, 7, 13, 19}
c. Domain: {5, 7, 13, 19} and Range: {0, 1, 2, 3
8. }
9. a.



b. The function is not invertible. Several ways to justify: the inverse fails the vertical line test; the original function fails the horizontal line test.

10. The function f is a horizontal line with equation $y = c$. The domain is the set of all real numbers, and the range is the single value c. Therefore the inverse would be a function whose domain is c and the range is all real numbers. This is the vertical line $x = c$. This is not a function. So $f(x) = c$ is not invertible.
11. a. $C(x) = 4x$
b. $C^{-1}x = \frac{1}{4}x$
c. The inverse function tells you the number of feet you bought, given the amount of money you spent.

Vocabulary

Inverse The inverse of a function is the relation obtained by interchanging the domain and range of a function.

Invertible A function is invertible if its inverse is a function.

One-to-one A function is one-to-one if every element of its domain is paired with exactly one element of its range.

5.2 Exponential Functions

Learning objectives

- Evaluate exponential expressions
- Identify the domain and range of exponential functions
- Graph exponential functions by hand and using a graphing utility
- Solve basic exponential equations

Introduction

In this lesson you will learn about **exponential functions,** a family of functions we have not studied in chapter 1 or chapter 2. In terms of the form of the equation, exponential functions are different from the other function families because the variable x is in the exponent. For example, the functions $f(x) = 2^x$ and $g(x) = 100(2)5^x$ are exponential functions. This kind of function can be used to model real situations, such as population growth, compound interest, or the decay of radioactive materials. In this lesson we will look at basic examples of these functions, and we will graph and solve exponential equations. This introduction to exponential functions will prepare you to study applications of exponential functions later in this chapter.

Evaluating Exponential Functions

Consider the function $f(x) = 2^x$. When we input a value for x, we find the function value by raising 2 to the exponent of x. For example, if $x = 3$, we have $f(3) = 2^3 = 8$. If we choose larger values of x, we will get larger function values, as the function values will be larger powers of 2. For example, $f(10) = 2^{10} = 1{,}024$.

Now lets consider smaller x values. If $x = 0$, we have $f(0) = 2^0 = 1$. If $x = -3$, we have $f(-3) = 2^{-3} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$. If we choose smaller and smaller x values, the function values will be smaller and smaller fractions. For example, if $x = -10$, we have $f(-10) = 2^{-10} = \left(\frac{1}{2}\right)^{10} = \frac{1}{1024}$. Notice that none of the x values we choose will result in a function value of 0. (This is the case because the numerator of the fraction will always be 1.) This tells us that while the domain of this function is the set of all real numbers, the range is limited to the set of positive real numbers. In the following example, you will examine the values of a similar function.

Example 1: For the function $g(x) = 3^x$, find $g(2)$, $g(4)$, $g(0)$, $g(-2)$, $g(-4)$.

Solution:

$g(2) = 3^2 = 9$

$g(4) = 3^4 = 81$

$g(0) = 3^0 = 1$

$g(-2) = 3^{-2} = \frac{1}{3^2} = \frac{1}{9}$

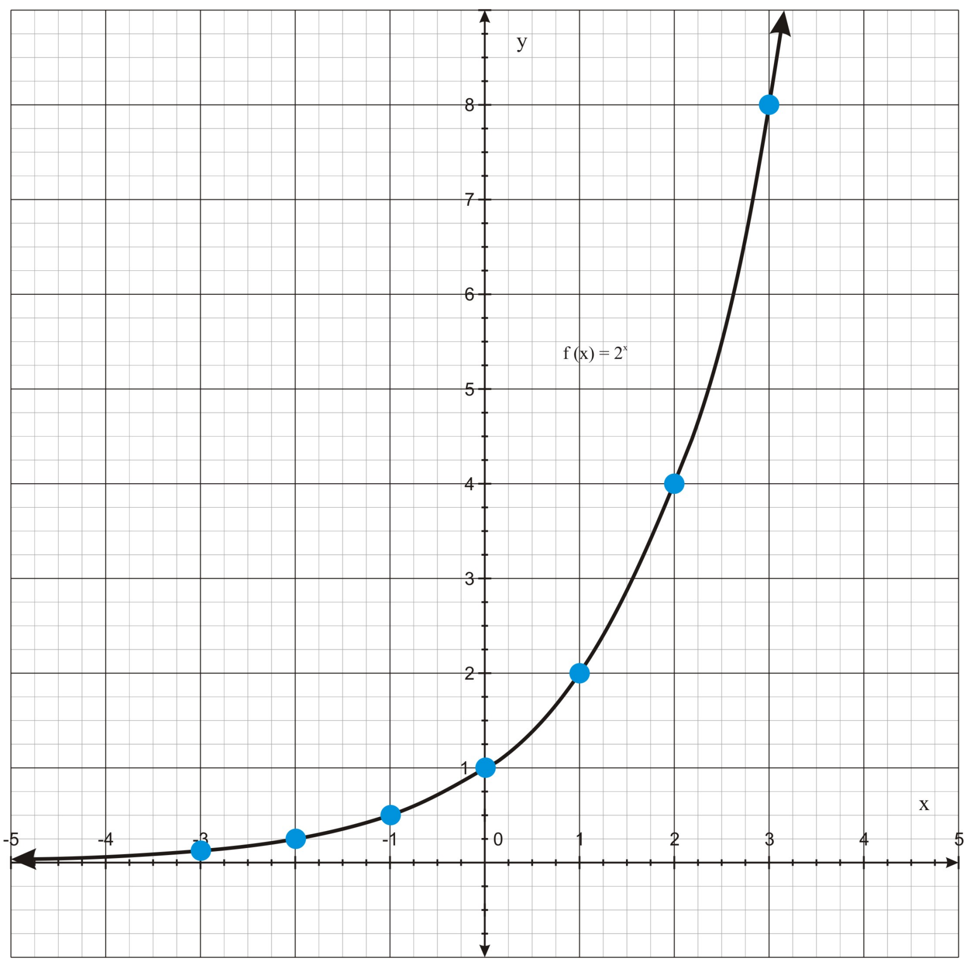
$g(-4) = 3^{-4} = \frac{1}{3^4} = \frac{1}{81}$

The values of the function $g(x) = 3^x$ behave much like those of $f(x) = 2^x$: if we choose larger values, we get larger and larger function values. If $x = 0$, the function value is 1. And, if we choose smaller and smaller x values, the function values will be smaller and smaller fractions. Also, the range of $g(x)$ is limited to positive values.

In general, if we have a function of the form $f(x) = a^x$, where a is a positive real number, the domain of the function is the set of all real numbers, and the range is limited to the set of positive real numbers. This restricted domain will result in a specific shape of the graph.

Graphing basic exponential functions

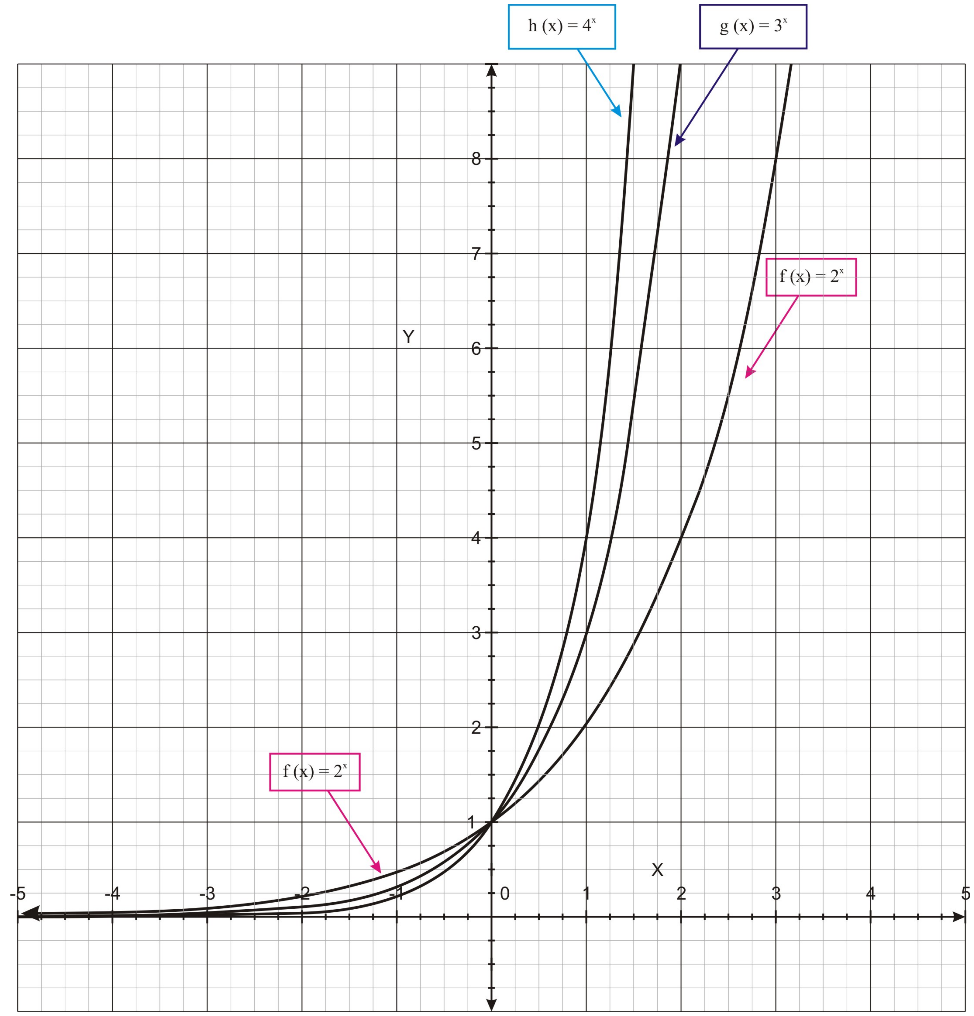
Lets now consider the graph of $f(x) = 2^x$. Above we found several function values, and we began to analyze the function in terms of large and small values of x. The graph below shows this function, with several points marked in blue.



Notice that as x approaches ∞, the function grows without bound. That is, $\lim_{x\to\infty}(2^x) = \infty$. However, if x approaches $-\infty$, the function values get closer and close to 0. That is, $\lim_{x\to-\infty}(2^x) = 0$. Therefore the function is asymptotic to the x-axis. This is the graphical result of the fact that the range of the function is limited to positive y values. Now lets consider the graph of $g(x) = 3^x$ and $h(x) = 4^x$.

Example 2: Use a graphing utility to graph $f(x) = 2^x$, $g(x) = 3^x$ and $h(x) = 4^x$. How are the graphs the same, and how are they different?

Solution: $f(x) = 2^x$, $g(x) = 3^x$ and $h(x) = 4^x$ are shown together below.



The graphs of the three functions have the same overall shape: they have the same end behavior, and they all contain the point (0, 1). The difference lies in their rate of growth. Notice that for positive x values, $h(x) = 4^x$ grows the fastest and $f(x) = 2^x$ grows the slowest. The function values for $h(x) = 4^x$ are highest and the function values for $f(x) = 2^x$ are the lowest for any given value of x. For negative x values, the relationship changes: $f(x) = 2^x$ has the highest function values of the three functions.

Now that we have examined these three parent graphs, we will graph using shifts, reflections, stretches and compressions.

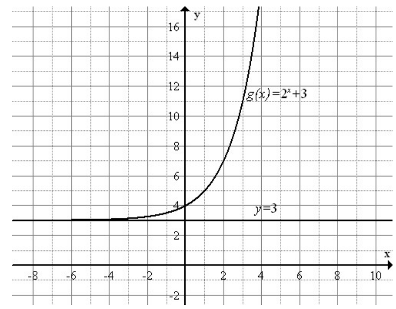
Graphing exponential functions using transformations.

Above we graphed the function $f(x) = 2^x$. Now lets consider a related function: $g(x) = 2^x + 3$. Every function value will be a power of 2, plus 3. The table below shows several values for the function:

TABLE 5.3:

x	$g(x) = 2^x + 3$
-2	$2^{-2} + 3 = \frac{1}{4} + 3 = 3\frac{1}{4}$
-1	$2^{-1} + 3 = \frac{1}{2} + 3 = 3\frac{1}{2}$
0	$2^0 + 3 = 1 + 3 = 4$
1	$2^1 + 3 = 2 + 3 = 5$
2	$2^2 + 3 = 4 + 3 = 7$
3	$2^3 + 3 = 8 + 3 = 11$

The function values follow the same kind of pattern as the values for $f(x) = 2^x$. However, because every function value is 3 more than a power of 2, the horizontal asymptote of the function is the line $y = 3$. The graph of this function and the horizontal asymptote are shown below.

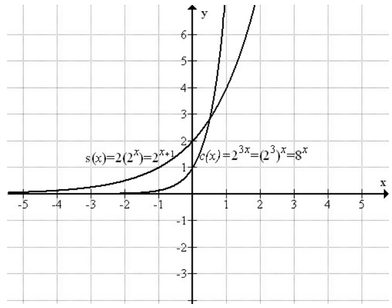


From your study of transformation of functions in chapter 1, you should recognize the graph of $g(x) = 2^x + 3$ as a vertical shift of the graph of $f(x) = 2^x$. In general, we can produce a graph of an exponential function with base 2 if we analyze the equation of the function in terms of transformations. The table below summarizes the different kinds of transformations of $f(x) = 2^x$. The issue of stretching will be discussed further below the table.

TABLE 5.4:

Equation	Relationship to $f(x)$=2^x	Range
$g(x) = \frac{2^x}{2^a} = 2^{x-a}$, for $a > 0$	Obtain a graph of g by shifting the graph of f *a* units to the right.	y >0
$g(x) = 2^a \cdot 2^x = 2^{a+x}$, for $a > 0$	Obtain a graph of g by shifting the graph of f *a* units to the left.	y >0
$g(x) = 2^x + a$, for $a > 0$	Obtain a graph of g by shifting the graph of f up *a* units.	y >a
$g(x) = 2^x - a$, for $a > 0$	Obtain a graph of g by shifting the graph of f down *a* units.	y >a
$g(x) = a(2^x)$, for $a > 0$	Obtain a graph of g by vertically stretching the graph of f *by a factor of a*.	y >0
$g(x) = 2^{ax}$, for $a > 0$	Obtain a graph of g by horizontally compressing the graph of f *by a factor of a*.	y >0
$g(x) = -2^x$	Obtain a graph of g by reflecting the graph of f over the x-axis.	y >0
$g(x) = 2^{-x}$	Obtain a graph of g by reflecting the graph of f over the y-axis.	y >0

As was discussed in chapter 1, a stretched graph can also be seen as a compressed graph. This is not the case for exponential functions because of the x in the exponent. Consider the function $s(x) = 2(2^x)$ and $c(x) = 2^{3x}$. The first function represents a vertical stretch of $f(x) = 2^x$ by a factor of 2. The second function represents a compression of $f(x) = 2^x$ by a factor of 3. The function c(x) is actually the same as another parent function: $c(x) = 2^{3x} = (2^3)^x = 8^x$. The function s(x) is actually the same as a shift of $f(x) = 2^x$: $s(x) = 2(2^x) = 2^1\ 2^x = 2^{x+1}$. The graphs of s and c are shown below. Notice that the graph of c has a y-intercept of 1, while the graph of s has a y-intercept of 2:

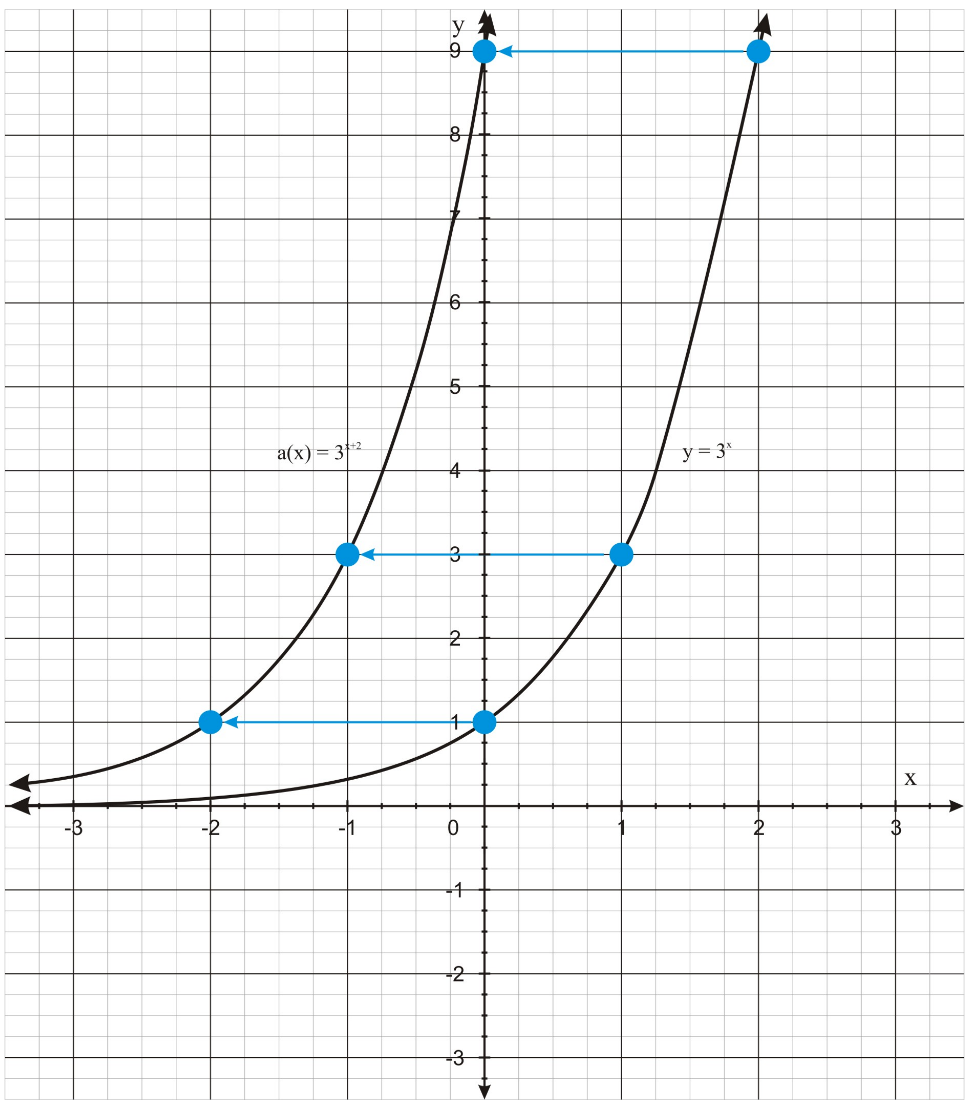


Example 3: Use transformations to graph the functions (a) $a(x) = 3^{x+2}$ and (b) $b(x) = -3^x + 4$

Solution:

a. $a(x) = 3^{x+2}$

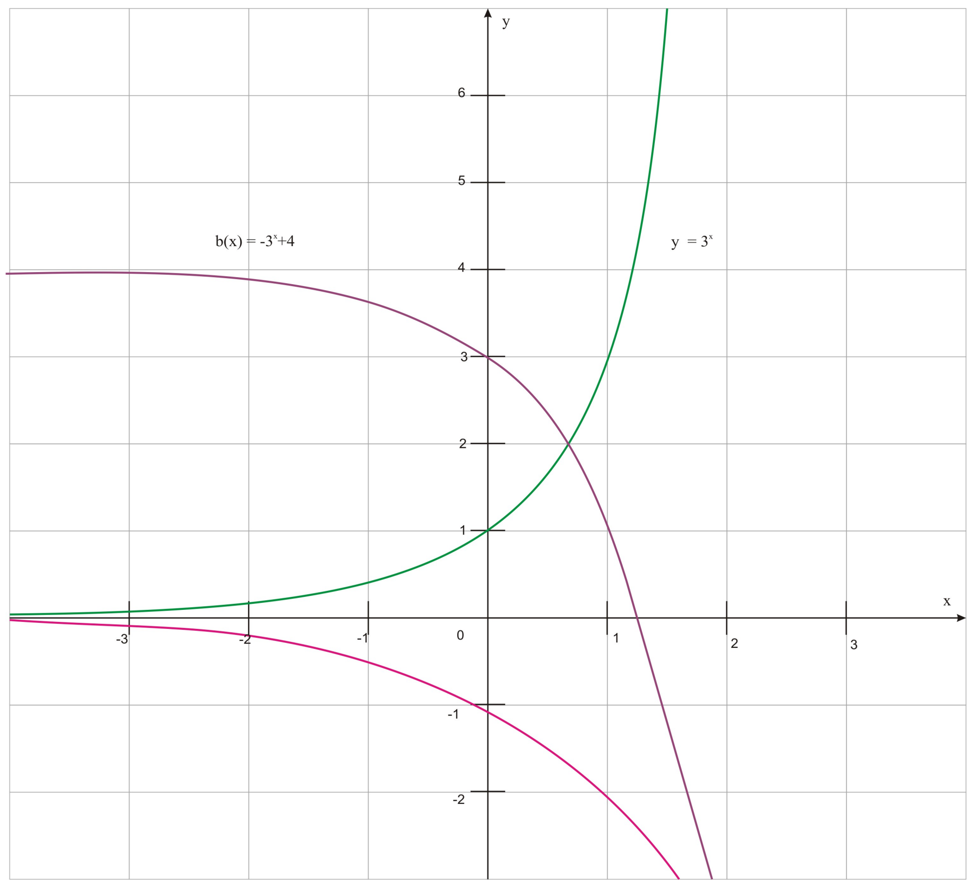
This graph represents a shift of $y = 3^x$ two units to the left. The graph below shows this relationship between the graphs of these two functions:



b. $b(x) = -3^x + 4$

This graph represents a reflection over the y-axis and a vertical shift of 4 units. You can produce a graph of $b(x)$

using three steps: sketch $y = 3^x$, reflect the graph over the x-axis, and then shift the graph up 4 units. The graph below shows this process:



While you can always quickly create a graph using a graphing utility, using transformations will allow you to sketch a graph relatively quickly on your own. If we start with a parent function such as $y=3^x$, you can quickly plot several points: (0, 1), (2, 9), (-1, 1/3), etc. Then you can transform the graph, as we did in the previous example.

Notice that when we sketch a graph, we choose x values, and then use the equation to find y values. But what if we wanted to find an x value, given a y value? This requires solving exponential equations.

Solving exponential equations

Solving an exponential equation means determining the value of x for a given function value. For example, if we have the equation $2^x = 8$, the solution to the equation is the value of x that makes the equation a true statement. Here, the solution is $x = 3$, as $2^3 = 8$.

Consider a slightly more complicated equation $3\ (2^{x\ +1}) = 24$. We can solve this equation by writing both sides of the equation as a power of 2:

$$3(2^{x+1}) = 24$$

$$\frac{3(2^{x+1})}{3} = \frac{24}{3}$$

$$2^{x+1} = 8$$

$$2^{x+1} = 2^3$$

To solve the equation now, recall a property of exponents: if $b^x = b^y$, then $x = y$. That is, if two powers of the same base are equal, the exponents must be equal. This property tells us how to solve:

$2^{x+1} = 2^3$

$\Rightarrow x+1 = 3$

$x = 2$

Example 4: Solve the equation $5^{6x+10} = 25^{x-1}$

Solution: Use the same technique as shown above:

$5^{6x+10} = 25^{x-1}$

$5^{6x+10} = (5^2)^{x-1}$

$5^{6x+10} = 5^{2x-2}$

$\Rightarrow 6x + 10 = 2x - 2$

$4x + 10 = -2$

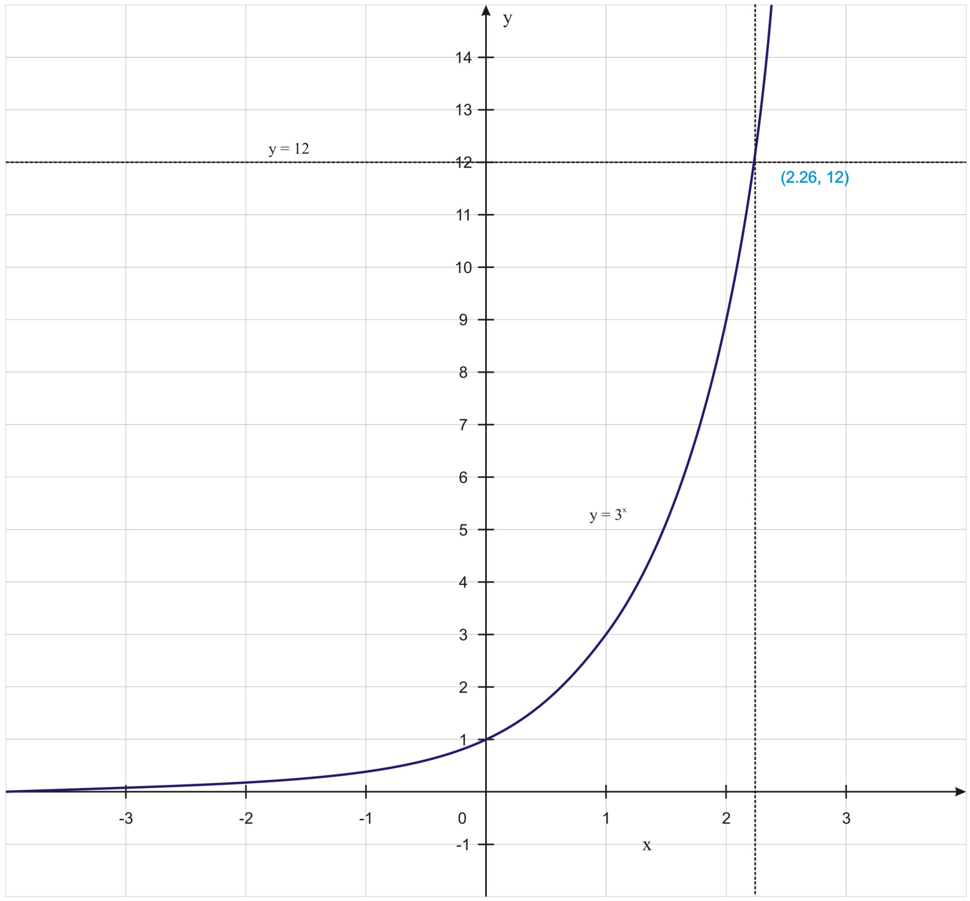
$4x = -12$

$x = -3$

In both of the examples of solving equations, it was possible to solve because we could write both sides of the equations as a power of the same exponent. But what if that is not possible?

Consider for example the equation $3^x = 12$. If you try to figure out the value of x by considering powers of 3, you will quickly discover that the solution is not a whole number. Later in the chapter we will study techniques for solving more complicated exponential equations. Here we will solve such equations using graphs.

Consider the function $y = 3^x$. We can find the solution to the equation $3^x = 12$ by finding the intersection of $y = 3^x$ and the horizontal line $y = 12$. Using a graphing calculators intersection capability, you should find that the solution is approximately $x = 2.26$.



Example 5: Use a graphing utility to solve each equation:

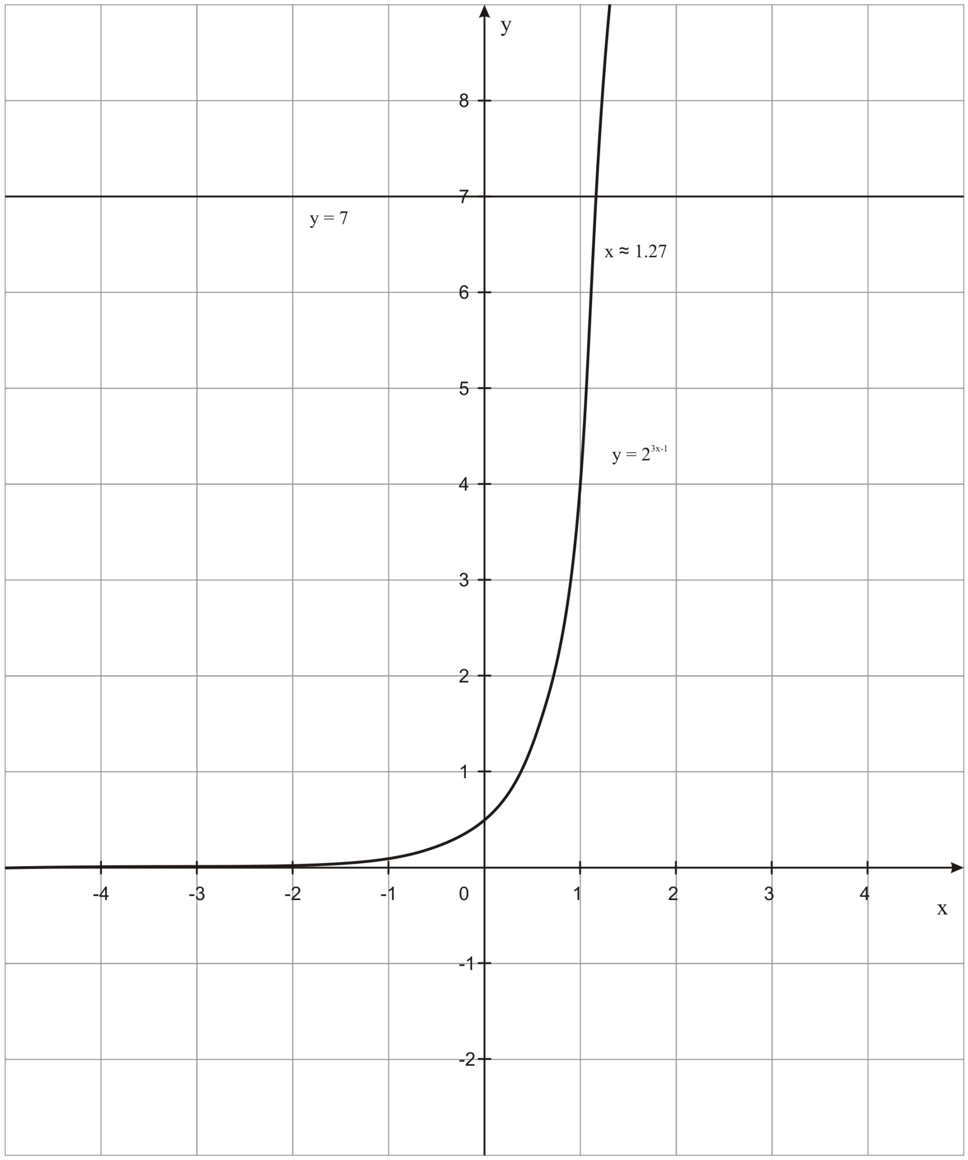
TABLE 5.5:

a. $2^{3x-1} = 7$	b. $6^{-4x} = 2^{8x-5}$

Solution:

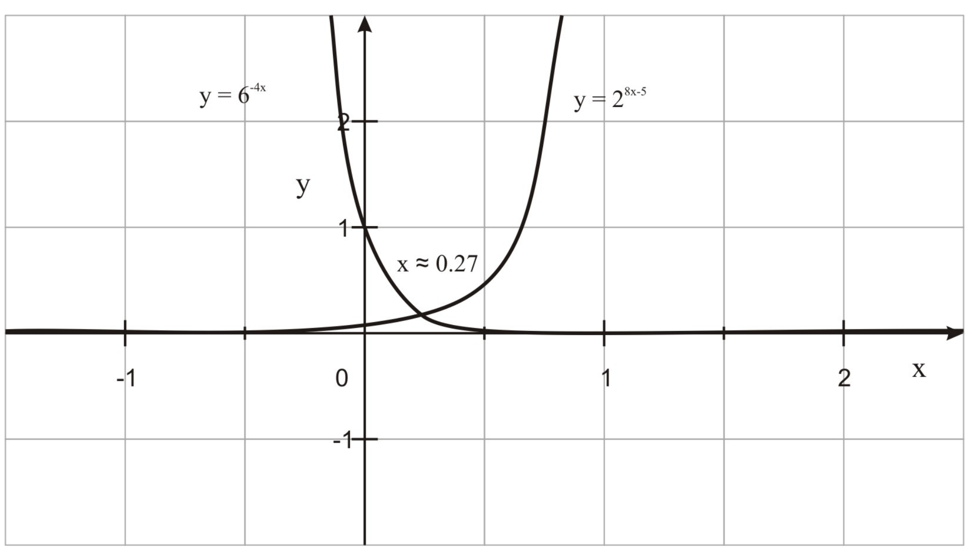
a. $2^{3x-1} = 7$

Graph the function $y = 2^{3x-1}$ and find the point where the graph intersects the horizontal line $y = 7$. The solution is x 1.27



b. $6^{-4x} = 2^{8x\ -5}$

Graph the functions $y = 6^{-4x}$ and $y = 2^{8x\ -5}$ and find their intersection point.



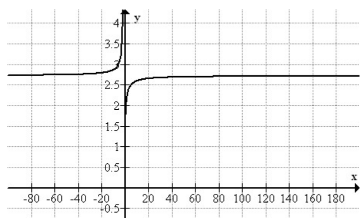
The solution is approximately x 0.27. (Your graphing calculator should show 9 digits: 0.272630365.)

In the examples we have considered so far, the bases of the functions have been positive integers. Now we will examine a sub-family of exponential functions with a special base: the number e.

The number *e* and the function $y = e^x$

In your previous studies of math, you have likely encountered the number . The number e is much like . First, both are **irrational** numbers: they cannot be expressed as fractions. Second, both numbers are **transcendental:** they are not the solution of any polynomial with rational coefficients.

Like , mathematicians found e to be a natural constant in the world. One way to discover e is to consider the function $f(x) = \left(1+\frac{1}{x}\right)^x$. The graph of this function is shown below.



Notice that as x approaches infinity, the graph of the function approaches a horizontal asymptote around $y = 2.7$. If you examine several function values, you will see that the limit is not exactly 2.7:

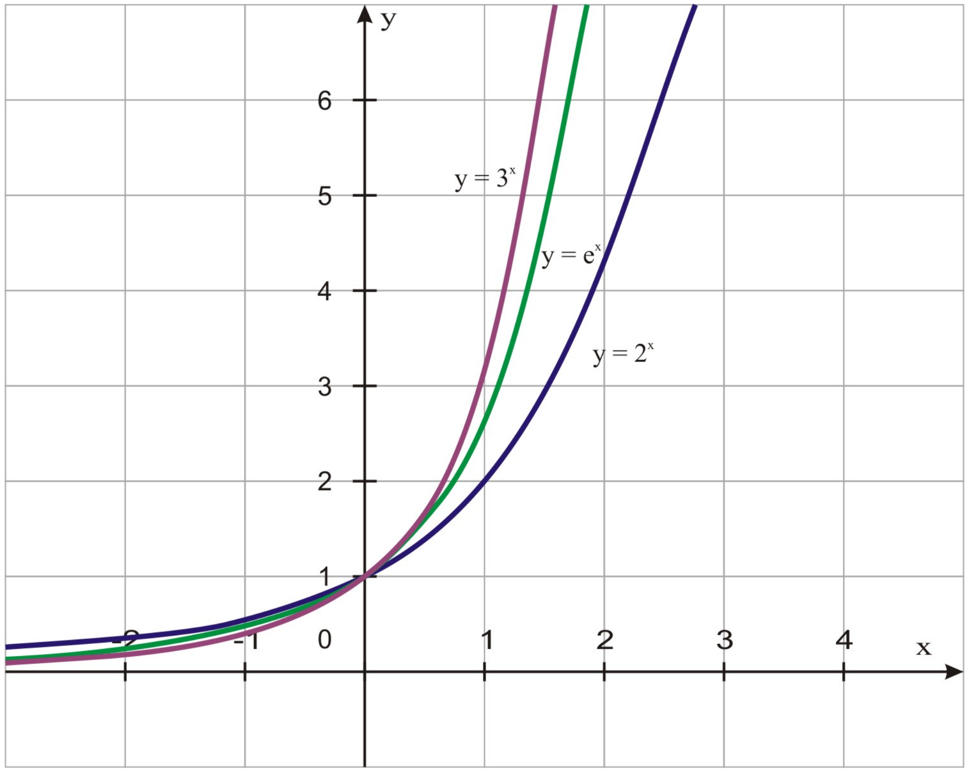
TABLE 5.6:

x	y
0	(not defined)
1	2
2	2.25
5	2.48832
10	2.5937424601
50	2.69158802907
100	2.70481382942
1000	2.71692393224
5000	2.7180100501
10,000	2.71814592683
50,000	2.7181825464614

Around $x = 100$, the function values pass 2.7, but they will never reach 2.8. The limit of the function as x approaches infinity is the constant e. The value of e is approximately 2.71828182845904523536. Again, like , we have to approximate the value of e because it is irrational.

The number e is used as the base of functions that can be used to model situations that involve growth or decay. For example, as you will learn later in the chapter, one method of calculating interest on a bank account or investment uses this number. Here we will examine the function $y = e^x$ in order to verify that its graph is similar to the other exponential functions we have graphed.

The graph below shows $y = e^x$, along with $y = 2^x$ and $y = 3^x$.



The graph of $y = e^x$ (in green) has the same shape as the graphs of the other exponential functions. It sits in between the graphs of the other two functions, and notice that the graph is closer to $y = 3^x$ than to $y = 2^x$. All three graphs have the same y-intercept: (0, 1). Thus the graph of this function is clearly a member of the same family, even though the base of the function is an irrational number.

Lesson Summary

This lesson has introduced the family of exponential functions. We have examined values of functions, towards understanding the behavior of graphs. In general, exponential functions have a horizontal asymptote, though one end of the function increases (or decreases, if it is a reflection) without bound.

In this lesson we have graphed these functions, solved certain exponential equations using our knowledge of exponents, and solved more complicated equations using graphing utilities. We have also examined the function $y = e^x$, which is a special member of the exponential family. In the coming lessons you will continue to learn about exponential functions, including the inverses of these functions, applications of these functions, and solving more complicated exponential equations using algebraic techniques.

Points to Consider

1. Why do exponential functions have horizontal asymptotes and not vertical asymptotes?
2. What would the graph of the inverse of an exponential function look like? What would its domain and range be?
3. How could you solve or approximate a solution to an exponential equation without using a graphing calculator?

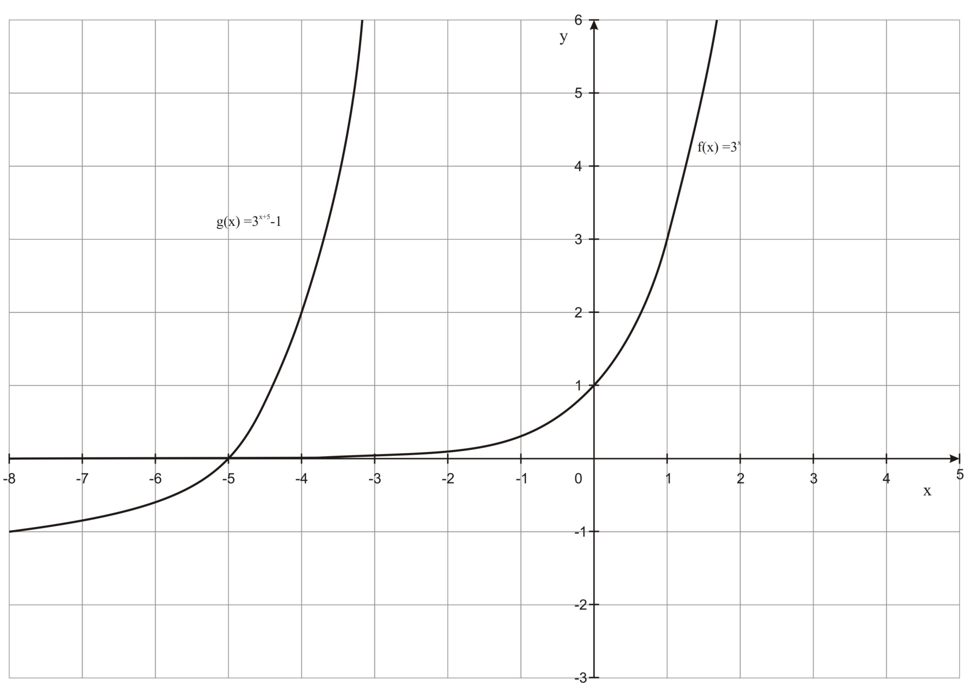
Review Questions

1. For the function $f(x) = 2^{3x-1}$, find $f(0)$, $f(2)$, and $f(-2)$.

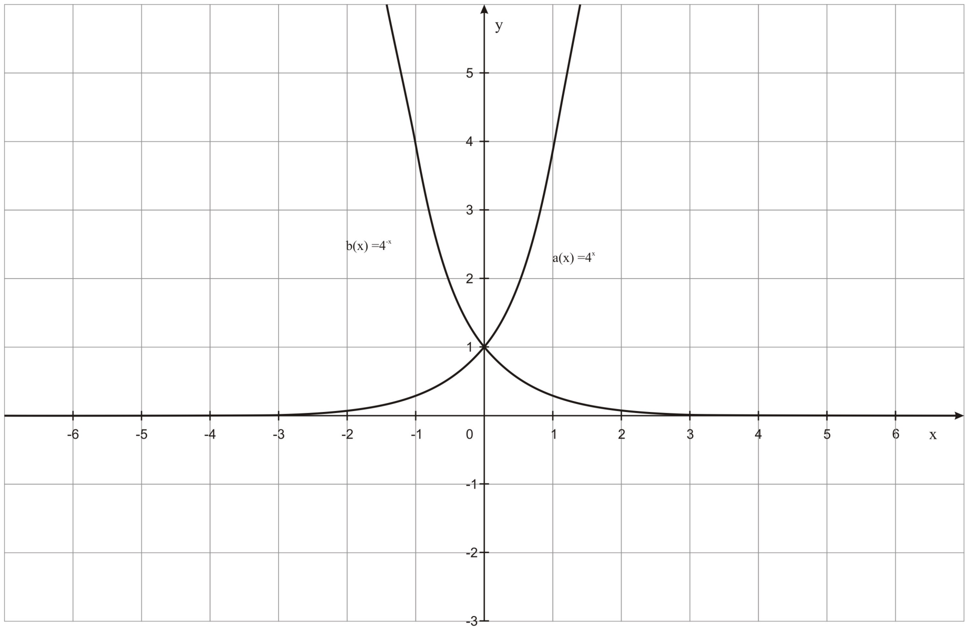
2. Graph the functions $f(x) = 3^x$ and $g(x) = 3^{x\ +5}$ -1. State the domain and range of each function.
3. Graph the functions $a(x) = 4^x$ and $b(x) = 4^{-x}$. State the domain and range of each function.
4. Graph the function $h(x) = -2^{x\ -1}$ using transformations. How is h related to $y = 2^x$?
5. Solve the equation: $5^{2x\ +1} = 25^{3x}$
6. Solve the equation: $4^{x2} + 1 = 16^x$
7. Use a graph to find an approximate solution to the equation $3^x = 14$
8. Use a graph to find an approximate solution to the equation $2^{-x} = 7^{2x\ +9}$
9. Sketch a graph of the function $f(x) = 3^x$ and its inverse. (Hint: You can graph the inverse by reflecting a function across the line y=x.) Is f one-to-one?
10. Consider the following situation: you inherited a collection of 125 stamps from a relative. You decided to continue to build the collection, and you vowed to double the size of the collection every year.
    a. Write an exponential function to model the situation. (The input of the function is the number of years since you began building the collection, and the output is the size of the collection.)
    b. Use your model to determine how long it will take to have a collection of 10,000 stamps.

Review Answers

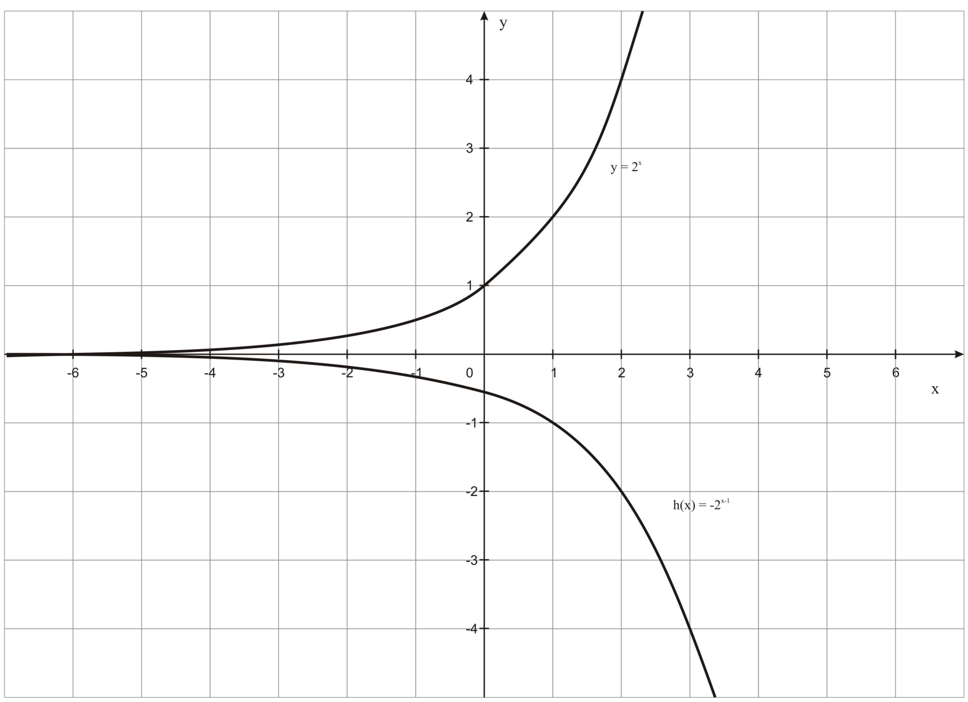
1. f(0)=1/2 , f(2)= 32, f(-2)= 1/128
2. The domain if both functions is the set of all real numbers.
   The range of f is the set of all real numbers 0.
   The range of g is the set of all real numbers -1



3. The domain of both functions is the set of all real numbers.
   The range of both functions is the set of all real numbers 0.

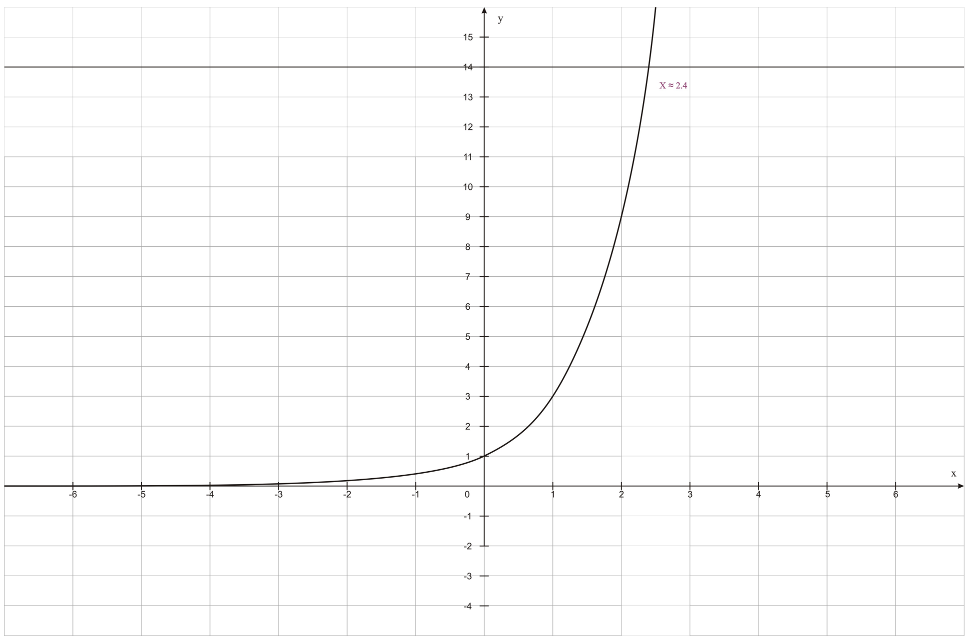


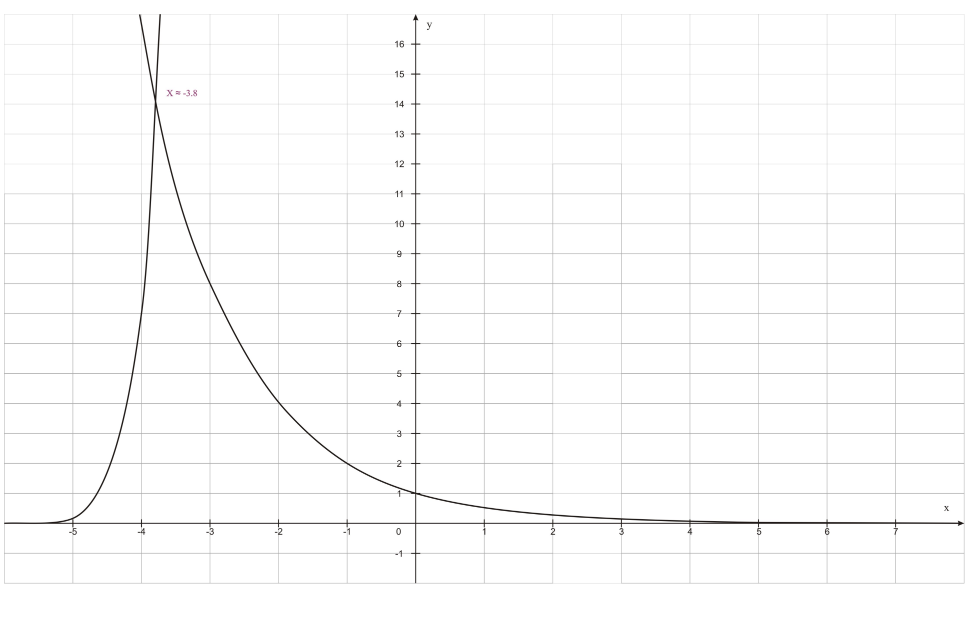
4. The function *h* represents a reflection over the *y* axis, and a horizontal shift 1 unit to the right.



5. $5^{2x+1} = 25^{3x}$
   $5^{2x+1} = 5^{6x}$
   $\Rightarrow 2x + 1 = 6x$
   $4x = 1$
   $x = 1/4$
6. $4^{x2} + 1 = 16^{x}$
   $4^{x2} + 1 = 4^{2x}$
   $\Rightarrow x^2 + 1 = 2x$
   $x^2 - 2x + 1 = 0$
   $(x - 1)(x - 1) = 0$
   $x = 1$

7. *x* 2.4

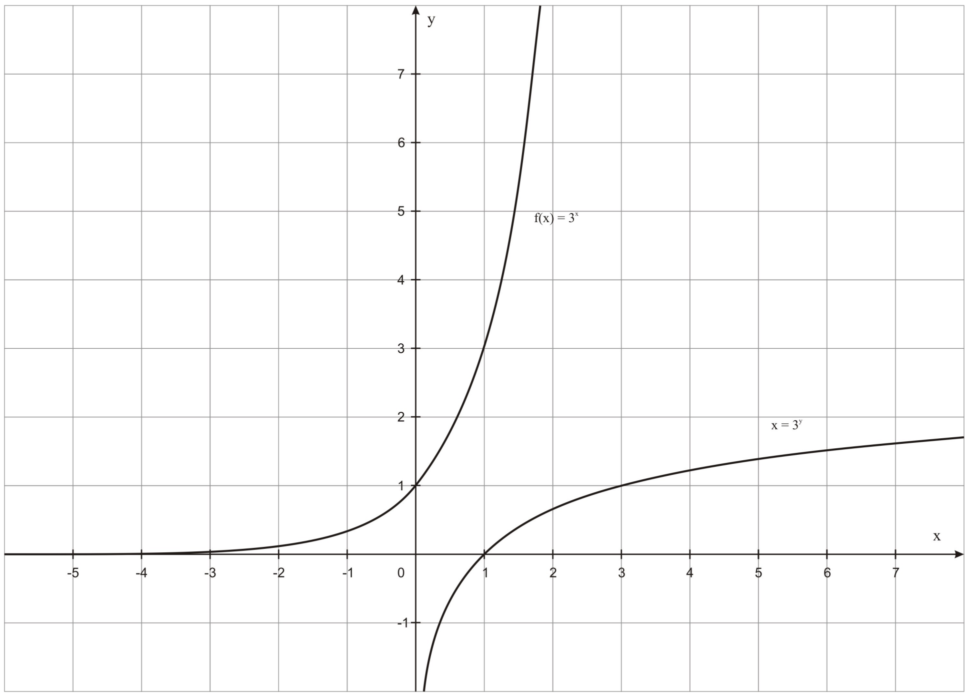




8.

x -3.8

9. *f* is a one-to-one function.



10. a. $S(t) = 125\left(2^t\right)$
    b. About 6.35 years

Vocabulary

e The number e is a transcendental number, often referred to as Eulers constant. Several mathematicians are credited with early work on e. Euler was the first to use this letter to represent the constant. The value of e is approximately 2.71828. The exact value is $\lim_{x \to \infty}\left(1+\frac{1}{x}\right)^x$.

Exponential Function An exponential function is a function for which the input variable x is in the exponent of some base b, where b is a real number.

Irrational number An irrational number is a number that cannot be expressed as a fraction of two integers.

Transcendental number A transcendental number is a number that is not a solution to any non-zero polynomial with rational roots.

The number is a transcendental number. It is the ratio of the circumference to the diameter in any circle.

5.3 Logarithmic Functions

Learning objectives

- Translate numerical and algebraic expressions between exponential and logarithmic form.
- Evaluate logarithmic functions.
- Determine the domain of logarithmic functions.
- Graph logarithmic functions.
- Solve logarithmic equations.

Introduction

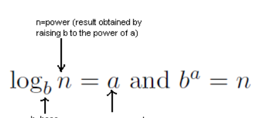
In the previous lesson we examined exponential expressions and functions. Now we will consider another representation for the same relationships involved in exponential expressions and functions.

Consider the function $y = 2^x$. Every x-value of this function is an **exponent**. Every y-value is a power of 2. As you learned in lesson 1, functions that are one-to-one have inverses that are functions. This is the case with exponential functions. If we take the inverse of $y = 2^x$ (by interchanging the domain and range) we obtain this equation: $x = 2^y$. In order to write this equation such that y is expressed as a function of x, we need a different notation.

The solution to this problem is found in the **logarithm**. John Napier originally introduced the logarithm to 17^{th} century mathematicians as a technique for simplifying complicated calculations. While todays technology allows us to do most any calculations we could imagine, logarithmic functions continue to be a focus of study in mathematics, as a useful way to work with exponential expressions and functions.

Changing Between Exponential and Logarithmic Expressions

Every exponential expression can be written in logarithmic form. For example, the equation $x=2^y$ is written as follows: $y=log_2 x$. In general, the equation $log_b n = a$ is equivalent to the equation $b^a=n$. That is, b is the **base**, a is the **exponent**, and n is the **power,** or the result you obtain by raising b to the power of a. Notice that the exponential form of an expression emphasizes the power, while the logarithmic form emphasizes the exponent. More simply put, a logarithm (or log for short) *is* an exponent.



We can write any exponential expression in logarithmic form.

Example 1: Rewrite each exponential expression as a log expression.

TABLE 5.7:

a. $3^4=81$	b. $b^{4x}=52$

Solution:

a. In order to rewrite an expression, you must identify its base, its exponent, and its power. The 3 is the base, so it is placed as the subscript in the log expression. The 81 is the power, and so it is placed after the log. Thus we have: 3^4=81 is the same as $log_3 81=4$ .To read this expression, we say the logarithm base 3 of 81 equals 4. This is equivalent to saying 3 to the 4^{th} power equals 81.

b. The b is the base, and the expression 4x is the exponent, so we have:$log_b 52=4x$. We say, log base b of 52, equals $4x$.

We can also express a logarithmic statement in exponential form.

Example 2: Rewrite the logarithmic expressions in exponential form.

TABLE 5.8:

a. $\log_{10} 100=2$	b. $\log_b w=5$

Solution:

a. The base is 10, and the exponent is 2, so we have: $10^2=100$ b. The base is b, and the exponent is 5, so we have: $b^5=w$.

Perhaps the most common example of a logarithm is the Richter scale, which measures the magnitude of an earthquake. The magnitude is actually the logarithm base 10 of the amplitude of the quake. That is, $m=\log_{10} A$. This means that, for example, an earthquake of magnitude 4 is 10 *times* as strong as an earthquake with magnitude 3. We can see why this is true of we look at the logarithmic and exponential forms of the expressions: An earthquake of magnitude 3 means $3=\log_{10} A$. The exponential form of this expression is $10^3=A$. Thus the amplitude of the quake is 1,000. Similarly, a quake with magnitude 4 has amplitude 10^4=10,000. We will return to this example in lesson 3.8.

Evaluating Logarithmic Functions

As noted above, a logarithmic function is the inverse of an exponential function. Consider again the function $y=2_x$ and its inverse $x=2^y$. Above, we rewrote the inverse as $y=\log_2 x$. If we want to emphasize the fact that the log equation represents a **function**, we can write the equation as $f(x)=\log_2 x$. To **evaluate** this function, we choose values of x and then determine the corresponding y values, or function values.

Example 3: Evaluate the function $f(x)=\log_2 x$ for the values:

TABLE 5.9:

a.x=2	b.x=1	c.x=-2

Solution:

a. If $x=2$, we have:

TABLE 5.10:

	$f(x)=\log_2 x$ $f(2)=\log_2 2$

To determine the value of $\log_2 2$, you can ask yourself: 2 to what power equals 2? Answering this question is often easy if you consider the exponential form: $2^?=2$

The missing exponent is 1. So we have $f(2)\text{-}\log_2 2=1$

b. If x=1 , we have:

TABLE 5.11:

	$f(x)$=log$_2x$ $f(1)$=log$_2$1

As we did in (a), we can consider the exponential form: $2^?$=1. The missing exponent is 0. So we have $f(1)$=log^{2}1=0.

c. If x=-2, we have:

TABLE 5.12:

$f(x)$	=log$_2x$
f(-2)	= log$_2$-2

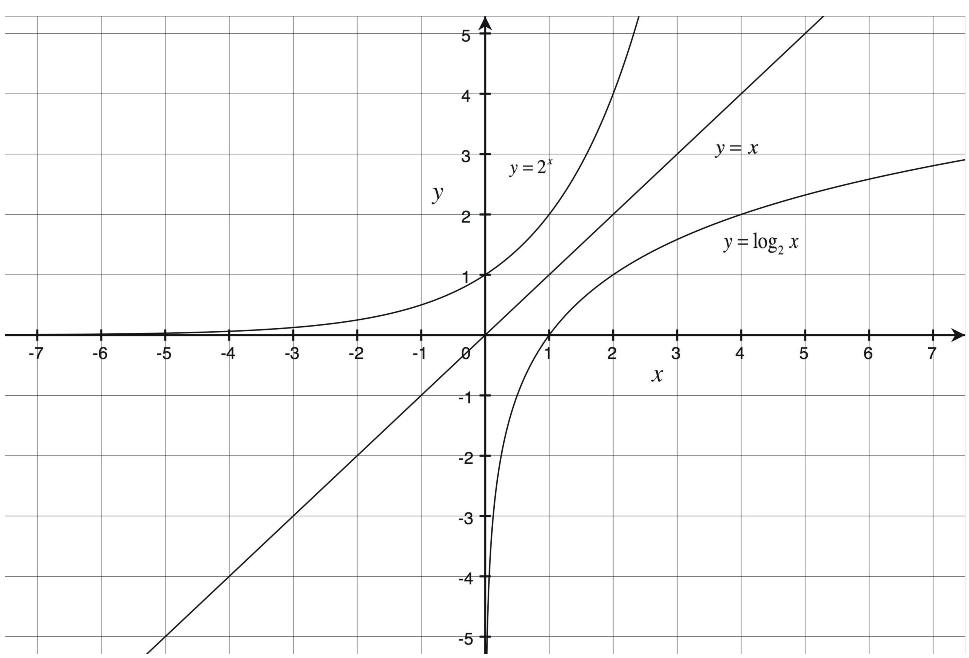
Again, consider the exponential form: $2^?$=-2. There is no such exponent. Therefore f(-2)=log$_2$-2 does not exist.

Example 3c illustrates an important point: there are restrictions on the **domain** of a logarithmic function. For the function $f(x)$=log$_2x$, x cannot be a negative number. Therefore we can state the domain of this function as: the set of all real numbers greater than 0. Formally, we can write it as a set: $\{x \in |x > 0\}$. In general, the domain of a logarithmic function is restricted to those values that will make the argument of the logarithm non-negative.

For example, consider the function $f(x)$=log$_3$(x-4). If you attempt to evaluate the function for x values of 4 or less, you will find that the function values do not exist. Therefore the domain of the function is $\{x \in |x > 4\}$. The domain of a logarithmic function is one of several key issues to consider when graphing.

Graphing Logarithmic Functions

Because the function $f(x)$=log$_2x$ is the inverse of the function $g(x)$=2^x, the graphs of these functions are reflections over the line $y = x$. The figure below shows the graphs of these two functions:



We can see that the functions are inverses by looking at the graph. For example, the graph of $g(x)$=2^x contains the point (1, 2), while the graph of $f(x)$=$\log_2$x contains the point (2, 1).

Also, note that while that the graph of $g(x)$=2^x is asymptotic to the x-axis, the graph of $f(x)$=$\log_2$x is asymptotic to the y-axis. This behavior of the graphs gives us a visual interpretation of the restricted range of g and the restricted domain of f.

When graphing log functions, it is important to consider x- values across the domain of the function. In particular, we should look at the behavior of the graph as it gets closer and closer to the asymptote. Consider $f(x)$=$\log_2$x for values of x between 0 and 1.

If x=1/2, then f(1/2)=$\log_2$(1/2)=-1 because 2^{-1}=1/2 If x=1/4, then f(1/4)=log_2(1/4)=-2 because 2^{-2}=1/4 If x=1/8}, then f(1/8)=$\log_2$(1/8)=-3 because 2^{-3}=1/8

From these values you can see that if we choose x values that are closer and closer to 0, the y values decrease (heading towards $-\infty$!). In terms of the graph, these values show us that the graph gets closer and closer to the y-axis. Formally we say that the vertical asymptote of the graph is $x = 0$.

Example 4: Graph the function $f(x)$=$\log_4 x$ and state the domain and range of the function.

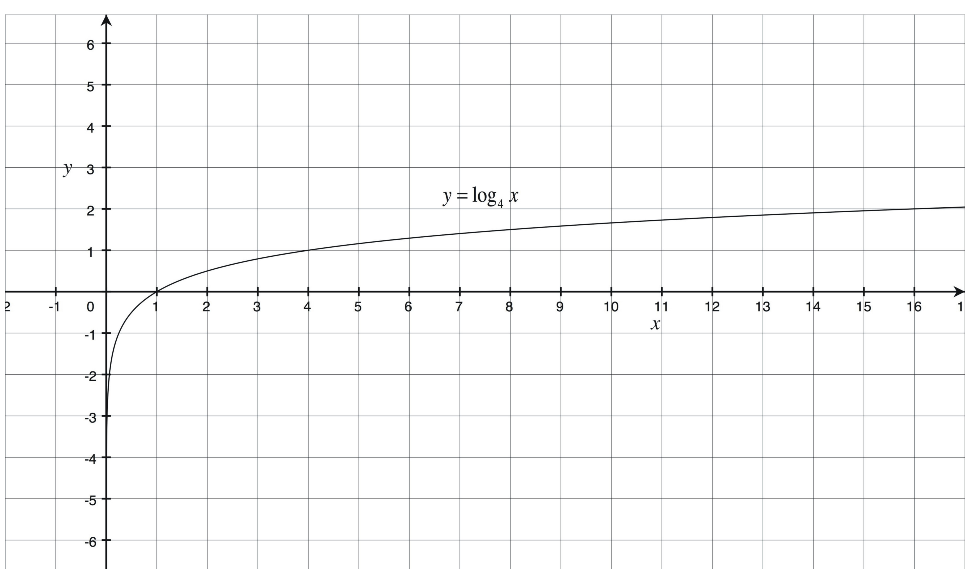
Solution: The function $f(x)$=$\log_4 x$ is the inverse of the function $g(x) = 4^x$. We can sketch a graph of $f(x)$ by evaluating the function for several values of x, or by reflecting the graph of g over the line $y = x$.

If we choose to plot points, it is helpful to organize the points in a table:

TABLE 5.13:

x	y=$\log_4 x$
1/4	
1	0
4	1
16	2

The graph is asymptotic to the y-axis, so the domain of f is the set of all real numbers that are greater than 0. We can write this as a set:$\{x \in |x > 0\}$. While the graph might look as if it has a horizontal asymptote, it does in fact continue to rise. The range is .



A note about graphing calculators: You can use a graphing calculator to graph logarithmic functions, but many calculators will only allow you to use base 10 or base e. However, after the next lesson you will be able to rewrite *any* log as a log with base 10 or base *e*.

In this section we have looked at graphs of logarithmic functions of the form $f(x) = \log_b x$. Now we will consider the graphs of other forms of logarithmic equations.

Graphing Logarithmic Functions Using Transformations

As you saw in the previous lesson, you can graph exponential functions by considering the relationships between equations. For example, you can use the graph of $f(x) = 2^x$ to sketch a graph of $g(x) = 2^x + 3$. Every y value of $g(x)$ is the same as a y value of $f(x)$, plus 3. Therefore we can shift the graph of $f(x)$ up 3 units to obtain a graph of $g(x)$.

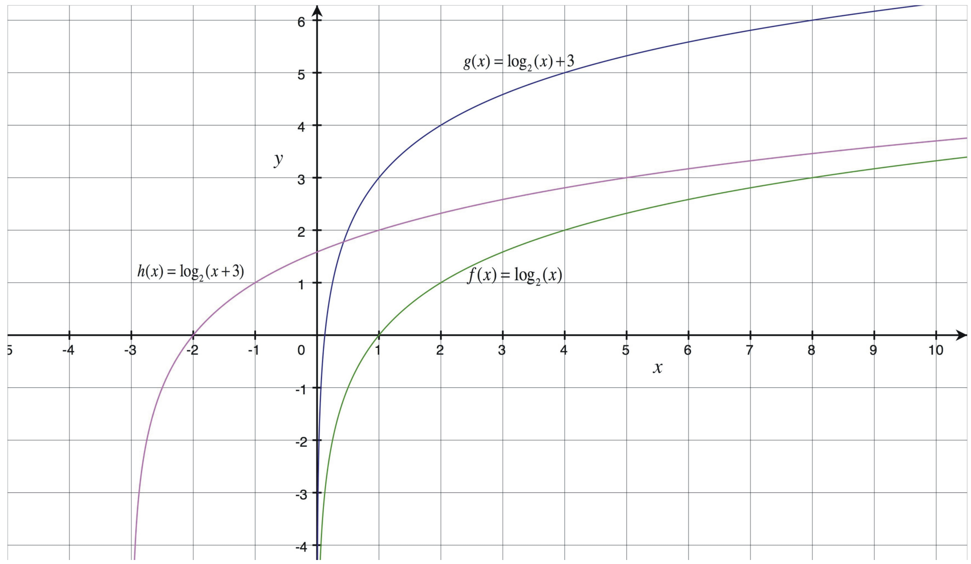
We can use the same relationships to efficiently graph log functions. Consider again the log function $f(x) = \log_2 x$. The table below summarizes how we can use the graph of this function to graph other related function.

TABLE 5.14:

Equation	Relationship to $f(x) = \log_2 x$	Domain
$g(x) = \log_2(x - a)$, for $a > 0$	Obtain a graph of g by shifting the graph of f a units to the right.	$x > a$
$g(x) = \log_2(x + a)$ for $a > 0$	Obtain a graph of g by shifting the graph of f a units to the left.	$x > -a$
$g(x) = \log_2(x) + a$ for $a > 0$	Obtain a graph of g by shifting the graph of f up a units.	$x > 0$
$g(x) = \log_2(x) - a$ for $a > 0$	Obtain a graph of g by shifting the graph of f down a units.	$x > 0$
$g(x) = a\log_2(x)$ for $a > 0$	Obtain a graph of g by vertically stretching the graph of f *by a factor of a*.	$x > 0$
$g(x) = -a\log_2(x)$, for $a > 0$	Obtain a graph of g by vertically stretching the graph of f *by a factor of a*, and by reflecting the graph over the x-axis.	$x > 0$
$g(x) = \log_2(-x)$	Obtain a graph of g by reflecting the graph of f over the y-axis.	$x < 0$

Example 5: Graph the functions $f(x) = \log_2(x)$, $g(x) = \log_2(x) + 3$, and $h(x) = \log_2(x + 3)$

Solution: The graph below shows these three functions together:



Notice that the location of the *3* in the equation makes a difference! When the 3 is added to $\log_2 x$, the shift is vertical. When the 3 is added to the *x*, the shift is horizontal. It is also important to remember that adding 3 to the *x* is a horizontal shift to the left. This makes sense if you consider the function value when $x = -3$:

$h(-3)=\log_2(-3 + 3)=\log_2 0$ = undefined

This is the vertical asymptote! To graph these functions, we evaluated them for certain values of *x*. But what if we want to know what the *x* value is for a particular *y* value? This means that we need to solve a logarithmic equation.

Solving Logarithmic Equations

In general, to solve an equation means to find the value(s) of the variable that makes the equation a true statement. To solve log equations, we have to think about what log *means*.

Consider the equation $\log_2 x=5$. What is the exponential form of this equation? The equation $\log_2 x=5$ means that $2^5 = x$. So the solution to the equation is $x = 2^5 = 32$.

We can use this strategy to solve many logarithmic equations.

Example 6: Solve each equation for *x*:

TABLE 5.15:

a. $\log_4 x = 3$	b. $\log_5(x + 1) = 2$	c. $1 + 2\log_3(x - 5) = 7$

Solution: a. Writing the equation in exponential form gives us the solution: $x = 4^3 = 64$.

b. Writing the equation in exponential form gives us a new equation: $5^2 = x + 1$.We can solve this equation for *x*:

TABLE 5.16:

5^2	$= x + 1$
25	$= x + 1$
x	$= 24$

c. First we have to isolate the log expression:

TABLE 5.17:

1 + 2$\log_3$(x-5)	= 7
2$\log_3$(x - 5)	= 6
$\log_3$(x-5)	= 3

Now we can solve the equation by rewriting it in exponential form:

TABLE 5.18:

$\log_3$(x-5)	= 3
3^3	= x - 5
27	= x - 5
x	= 32

We can also solve equations in which both sides of the equation contain logs. For example, consider the equation $\log_2$(3x-1)=$\log_2$(5x - 7). Because the logarithms have the same base (2), the arguments of the log (the expressions 3x - 1 and 5x - 7) *must be equal*. So we can solve as follows:

TABLE 5.19:

$\log_3$(3x-1)	= $\log_2$(5x - 7)
3x - 1	= 5x - 7
+7 |+7	
3x + 6	= 5x
-3x	
6	= 2x
x	= 3

Example 7: Solve for x: $\log_2$(9x)=$\log_2$(3x + 8)

Solution: The log equation implies that the expressions 9x and 3x + 8 are equal:

TABLE 5.20:

$log_2(9x)$	$= log_2(3x+8)$
$9x$	$= 3x+8$
$-3x$	$-3x$
$6x$	$= 8$
x	$= \frac{8}{6}$
x	$= \frac{4}{3}$

Lesson Summary

In this lesson we have defined the logarithmic function as the inverse of the exponential function. When working with logarithms, it helps to keep in mind that a *logarithm is an exponent*. For example, 3 = $\log_2 8$ and 2^3 = 8 are two forms of the same numerical relationship among the three numbers 2, 3, and 8. The 2 is the base, the 3 is the exponent, and 8 is the 3^{rd} power of 2.

Because logarithmic functions are the inverses of exponential functions, we can use our knowledge of exponential functions to graph logarithmic functions. You can graph a log function either by reflecting an exponential function over the line $y = x$, or by evaluating the function and plotting points. In this lesson you learned how to graph parent graphs such as $y = \log_2 x$ and $y = \log_4 x$, as well as how to use these parent graphs to graph more complicated log functions. When graphing, it is important to keep in mind that logarithmic functions have restricted domains. Each graph will have a vertical asymptote.

We can also use our knowledge of exponential relationships to solve logarithmic equations. In this lesson we solved 2 kinds of logarithmic equations. First, we solved equations by rewriting the equations in exponential form. Second, we solved equations in which both sides of the equation contained a log. To solve these equations, we used the following rule:

$\log_b f(x) = \log_b g(x)\, f(x) = g(x)$.

Points to Consider

1. What methods can you use to graph logarithmic functions?
2. What methods can you use to solve logarithmic equations?
3. What forms of log equations can you solve using the methods in this lesson? Can you write an equation that cannot be solved using these methods?

Review Questions

Write the exponential statement in logarithmic form.

1. $3^2 = 9$
2. $z^4 = 10$
3. Write the logarithmic statement in exponential form.
4. $\log_5 25 = 2$
5. $log_4 \frac{1}{6} = -1$
6. Complete the table of values for the function $f(x) = \log_3 x$

TABLE 5.21:

x	y = f(x)
1/9	
1/3	
1	
3	
9	

7.
8. Use the table above to graph $f(x) = \log_3 x$. State the domain and range of the function.
9. Consider $g(x) = -\log_3(x - 2)$
   a. How is the graph of $g(x)$ related to the graph of $f(x) = \log_3 x$?
   b. Graph $g(x)$ by transforming the graph of $f(x)$.
10. Solve each logarithmic equation:
    a. $\log_3 9x = 4$
    b. $7 + \log_2 x = 11$ (Hint: subtract 7 from both sides first.)
11. Solve each logarithmic equation:

TABLE 5.22:

a. $\log_5 6x = -1$	b. $\log_5 6x = \log_5(2x + 16)$	c. $\log_5 6x = \log_5(3x - 10)$

12.
13. Explain why the equation in 9c has no solution.

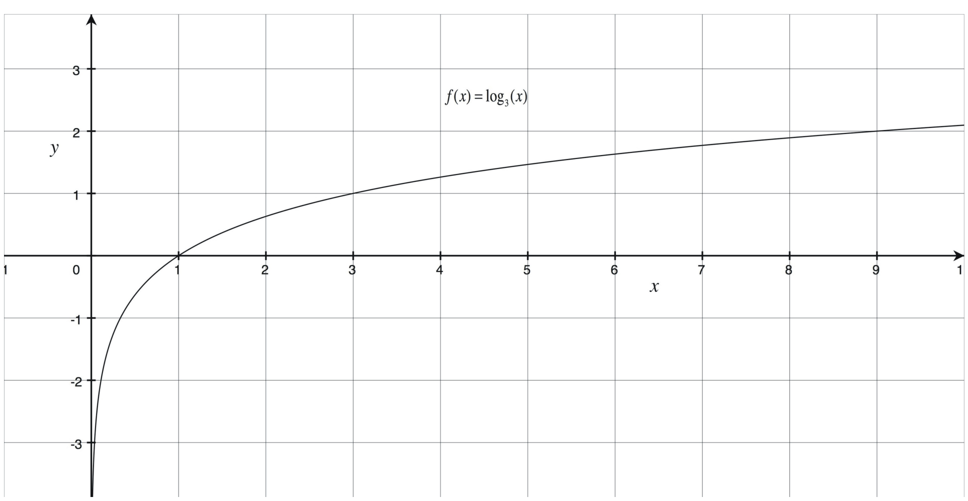
Review Answer

1. $3^2 = 9$
2. $z^4 = 10$
3. $5^2 = 25$
4. $6^{-1} = \frac{1}{6}$

TABLE 5.23:

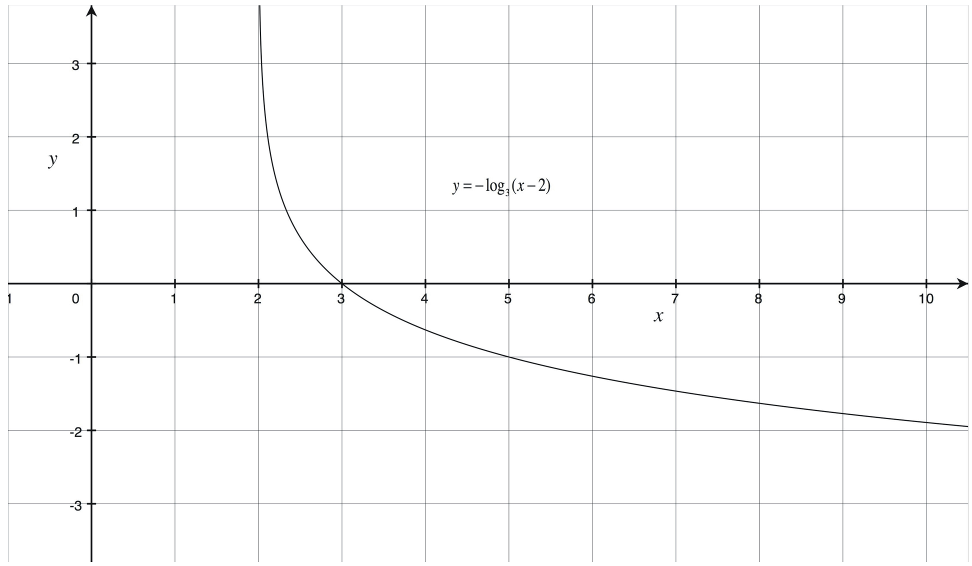
x	y = f(x)
1/9	
1/3	
1	0
3	1
9	2

5.



6.
D: All real numbers >0
R: All real numbers.

7. a. The graph of g(x) can be obtained by shifting the graph of f(x) 2 units to the right, and reflecting it over the x- axis.
b.



8. The solutions are:

TABLE 5.24:

a. x = 9	b. x = 16

9.
10. The solutions are:

TABLE 5.25:

a. x = 1/30	b. x = 4	c. no solution

11.
12. When we solve 6*x*=3*x*-10 we find that *x*=-10/3, a value outside of the domain. Because there is no other *x* value that satisfies the equation, there is no solution.

Vocabulary

Argument The expression inside a logarithmic expression. The argument represents the power in the exponential relationship.

Asymptote An asymptote is line whose distance to a given curve tends to zero. An asymptote may or may not intersect its associated curve.

Domain The domain of a function is the set of all values of the independent variable (x) for which the function is defined.

Evaluate To evaluate a function is to identify a function value (y) for a given value of the independent variable (x).

Function A function is a relation between a domain (set of x values) and range (set of y values) in which every element of the domain is paired with one and only one element of the range. A function that is one to one is a function in which every element of the domain is paired with exactly one y value.

Logarithm The exponent of the power to which a base number must be raised to equal a given number.

Range The range of a function is the set of all function values, or values of the dependent variable (y).

5.4 Properties of Logarithms

Learning objectives

- Use properties of logarithms to write logarithmic expressions in different forms.
- Evaluate common logarithms and natural logarithms.
- Use the change of base formula and a scientific calculator to find the values of logs with any bases.

Introduction

In the previous lesson we defined the logarithmic function as the inverse of an exponential function, and we evaluated log expressions in order to identify values of these functions. In this lesson we will work with more complicated log expressions. We will develop properties of logs that we can use to write a log expression as the sum or difference of several expressions, or to write several expressions as a single log expression. We will also work with logs with base 10 and base e, which are the bases most often used in applications of logarithmic functions.

Properties of Logarithms

Because a logarithm is an exponent, the properties of logs are the same as the properties of exponents. Here we will prove several important properties of logarithms.

Property 1: $\log_b(xy) = \log_b x + \log_b y$

Proof: Let $\log_b x = n$ and $\log_b y = m$.

Rewrite both log expressions in exponential form:$\log_b x = n \; b^n = x$

$\log_b y = m \; b^m = y$

Now multiply x and y: $xy = b^n \; b^m = b^{n+m}$ Therefore we have an exponential statement: $b^{n+m} = xy$. The log form of the statement is: $\log_b xy = n + m$. Now recall how we defined n and m:

$\log_b xy = n + m = \log_b \text{x} + log_b \; y.$

Property 2: $log_b \left(\frac{x}{y}\right) = log_b x - log_b y$

We can prove property 2 analogously to the way we proved property 1.

Proof: Let $\log_b x = n$ and $\log_b y = m$.

Rewrite both log expressions in exponential form:$\log_b x = n \; b^n = x$

$\log_b y = m \; b^m = y$

Now divide x by y: $\frac{x}{y} = \frac{b^n}{b^m} = b^{n-m}$ Therefore we have an exponential statement: $b^{n-m} = \frac{x}{y}$. The log form of the statement is: $log_b \left(\frac{x}{y}\right) = n - m$. Now recall how we defined n and m:

$log_b \left(\frac{x}{y}\right) = n - m = log_b x - log_b y.$

Property 3: $\log_b x^n = n \log_b x$

The proof of the third property relies on another property of logs that we can derive by thinking about the definition of a log. Consider the expression $\log_2 2^{13}$. What does this expression *mean?*

The exponential form of $\log_2 2^{13} = ?$ is $2^? = 2^{13}$. Looking at the exponential form should convince you that the missing exponent is 13. That is, $\log_2 2^{13} = 13$. In general, $\log_b b^n = n$. This property will be used in the proof of property 3.

Proof (of Property 3):

Let $\log_b x = w$. The exponential form of this log statement is $b^w = x$. If we raise both sides of this equation to the power of n, we have $(b^w)^n = x^n$.

Using the power property of exponents, this equation simplifies to $b^{wn} = x^n$ If two expressions are equal, then the logs of both expressions are equal:

$$\log_b \mathrm{b}^{wn} = \log_b x^n$$

Now consider the value of the left side of the equation: $\log_b b^{wn}$= wn. Above, we showed that $b^{wn} = x^n$. By substitution, we have $\log_b x^n = wn$. Above, we defined w: $\log_b x = w$. By substitution, we have

$$\log_b x^n = (\log_b x)\, n = n\log_b x.$$

We can use these properties to rewrite log expressions.

Expanding expressions

Using the properties we have derived above, we can write a log expression as the sum or difference of simpler expressions. Consider the following examples:

1. $\log_2 8x = \log_2 8 + \log_2 x = 3 + \log_2 x$
2. $log_3\left(\frac{x^2}{3}\right) = log_3 x^2 - log_3 3 = 2log_3 x - 1$

Using the log properties in this way is often referred to as "expanding". In the first example, expanding the log allowed us to simplify, as log2 8 = 3. Similarly, in the second example, we simplified using the log properties, and the fact that $\log_3 3 = 1$.

Example 1: Expand each expression:

TABLE 5.26:

a. $log_5 25x^2y$	b. $log_{10}\left(\frac{100x}{9b}\right)$

Solution:

a. $\log_5 25x^2y = \log_5 25 + \log_5 x^2 + \log_5 y = 2 + 2\log_5 x + \log_5 y$ b.

TABLE 5.27:

$log_{10}\left(\frac{100x}{9b}\right)$	$= log_{10}100x - log_{10}9b$ $= log_{10}100 + log_{10}x - [log_{10}9 + log_{10}b]$ $= 2 + log_{10}x - log_{10}9 - log_{10}b$

Just as we can write a single log expression as a sum and difference of expressions, we can also write expanded expressions as a single expression.

Condensing expressions

To condense a log expression, we will use the same properties we used to expand expressions. Consider the expression $\log_6 8 + \log_6 27$. Alone, each of these expressions does not have an integer value. The value of $\log_6 8$ is between 1 and 2; the value of $\log_6 27$ is also between 1 and 2. If we condense the expression, we get:

$\log_6 8 + \log_6 27 = \log_6 (8\ 27) = \log_6 216 = 3$

We can also condense algebraic expressions. This will be useful later for solving logarithmic equations.

Example 2: Condense each expression:

TABLE 5.28:

a. $2\log_3 x + \log_3 5x - \log_3 (x+1)$	b. $\log_2 (x^2 - 4) - \log_2 (x+2)$

Solution:

a. $$2log_3x+log_35x-log_3(x+1) = log_3x_2+log_35x-log_3(x+1)$$
$$= log_3(x^2(5x))-log_3(x+1)$$
$$= log_3\left(\frac{5x^3}{x+1}\right)$$

b. $$log_2(x^2-4)-log_2(x+2) = log_2\left(\frac{x^2-4}{x+2}\right)$$
$$= log_2\left(\frac{(x+2)(x-2)}{x+2}\right)$$
$$= log_2(x-2)$$

It is important to keep in mind that a log expression may not be defined for certain values of x. First, the argument of the log must be positive. For example, the expressions in example 2b above are not defined for x 2 (which allows us to "cancel" (x+2) without worrying about the condition x -2).

Second, the argument must be defined. For example, in example 2a above, the expression $\left(\frac{5x^3}{x+1}\right)$ is undefined if x = -1.

The log properties apply to logs with any real base. Next we will examine logs with base 10 and base e, which are the most common bases for logs (though only one is actually called common).

Common logarithms and natural logs

A **common logarithm** is a log with base 10. We can evaluate a common log just as we evaluate any other log. A common log is usually written without a base.

Example 3: Evaluate each log

TABLE 5.29:

a. log 1	b. log 10	c. $log\sqrt{10}$

Solution:

a. log 1 = 0, as $10^0 = 1$. b. log 10 = 1, as $10^1 = 10$ c. $log\sqrt{10} = \frac{1}{2}$ because $\sqrt{10} = 10^{1/2}$

As noted in lesson 3, logarithms were introduced in order to simplify calculations. After Napier introduced the logarithm, another mathematician, Henry Briggs, proposed that the base of a logarithm be standardized as 10. Just as Napier had labored to compile tables of log values (though his version of the logarithm is somewhat different from what we use today), Briggs was the first person to publish a table of common logs. This was in 1617!

Until recently, tables of common logs were included in the back of math textbooks. Publishers discontinued this practice when scientific calculators became readily available. A scientific calculator will calculate the value of a common log to 8 or 9 digits. Most calculators have a button that says LOG. For example, if you have TI graphing calculator, you can simply press LOG, and then a number, and the calculator will give you a log value up to 8 or 9 decimal places. For example, if you enter LOG(7), the calculator returns .84509804. This means that $10^{.84509804}$ 7. If we want to judge the reasonableness of this value, we need to think about powers of 10. Because $10^1 = 10$, log(7) should be less than 1.

Example 4: For each log value, determine two integers between which the log value should lie. Then use a calculator to find the value of the log.

TABLE 5.30:

a. log 50	b. log 818

Solution:

a. log 50The value of this log should be between 1 and 2, as $10^1 = 10$, and $10^2 = 100$. Using a calculator, you should find that log 50 1.698970004.

b. log 818

The value of this log should be between 2 and 3, as $10^2 = 100$, and $10^3 = 1000$.

Using a calculator, you should find that log 818 2.912753304.

The calculators ability to produce log values is an example of the huge benefit that technology can provide. Only a few years ago, the calculations in the previous example would have each taken several minutes, while now they only take several seconds. While most people might not calculate log values in their every day lives, scientists and engineers are grateful to have such tools to make their work faster and more efficient.

Along with the LOG key on your calculator, you will find another logarithm key that says LN. This is the abbreviation for the **natural log**, the log with base e. Natural logs are written using ln instead of log. That is, we write the expression $\log_e x$ as In x. How you evaluate a natural log depends on the argument of the log. You can evaluate some natural log expressions without a calculator. For example, ln $e = 1$, as $e^1 = e$. To evaluate other natural log expressions requires a calculator. Consider for example ln 7. Recall that e 2.7. This tells us that ln 7 should be slightly less than 2, as $(2.7)^2 = 7.29$. Using a calculator, you should find that ln 7 1.945910149.

Example 5: Find the value of each natural log.

TABLE 5.31:

a. $ln100$	b. $ln\sqrt{e}$

Solution:

a. ln 100 is between 4 and 5. You can estimate this by considering powers of 2.7, or powers of 3: $3^4 = 81$, and $3^5 = 243$.Using a calculator, you should find that ln 100 4.605171086.

b. Recall that a square root is the same as an exponent of 1/2. Therefore $ln\sqrt{e} = ln(e^{1/2}) = 1/2$

You may have noticed that the common log and the natural log are the only log buttons on your calculator. We can use either the common log or the natural log to find the values of logs with other bases.

Change of Base

Consider the log expression $\log_3 35$. The value of this expression is approximately 3 because $3^3 = 27$. In order to find a more exact value of $\log_3 35$, we can rewrite this expression in terms of a common log or natural log. Then we can use a calculator.

Lets consider a general log expression, $\log_b x = y$. This means that $b^y = x$. Recall that if two expressions are equal, then the logs of the expressions are equal. We can use this fact, and the power property of logs, to write $\log_b x$ in terms of common logs.

TABLE 5.32:

$b^y = x \Rightarrow log b^y = log x$	The logs of the expressions are equal

TABLE 5.33:

$\Rightarrow y log b = log x$	Use the power property of logs
$\Rightarrow y = \frac{log x}{log b}$	Divide both sides by $log b$
$\Rightarrow log_b x = \frac{log x}{log b}$	Substitute $log_b x = y$

The final equation, $log_b x = \frac{log x}{log b}$, is called the change of base formula. Notice that the proof did not rely on the fact that the base of the log is 10. We could have used a natural log. Thus another form of the change of base formula is $\log_b x = \frac{ln x}{ln b}$.

Note that we could have used a log with any base, but we use the common log and the natural log so that we can use a calculator to find the value of an expression. Consider again $\log_3 35$. If we use the change of base formula, and then a calculator, we find that

$$log_3 35 = \frac{log 35}{log 3} = 3.23621727.$$

Example 6: Estimate the value, and then use the change of base formula to find the value of $\log_2 17$.

Solution: $\log_2 17$ is close to 4 because $2^4 = 16$ and $2^5 = 32$. Using the change of base formula, we have $log_2 17 = \frac{log 17}{log 2}$. Using a calculator, you should find that the approximate value of this expression is 4.087462841.

Lesson Summary

In this lesson we have developed and used properties of logarithms, including a formula that allows us to calculate the value of a log expression with any base. Out of context, it may seem difficult to understand the value of these kind of calculations. However, as you will see in later lessons in this chapter, we can use exponential and logarithmic functions to model a variety of phenomena.

Points to Consider

1. Why is the common log called common? Why 10?
2. Why would you want to estimate the value of a log before using a calculator to find its exact value?
3. What kind of situations might be modeled with a logarithmic function?

Review Questions

1. Expand the expression: $\log_b 5x^2$
2. Expand the expression: $\log_3 81x^5$
3. Condense the expression: $\log(x + 1) + \log(x - 1)$
4. Condense the expression: $3\ln(x) + 2\ln(y) - \ln(5x - 2)$
5. Evaluate the expressions:
   a. log 1000
   b. log 0.01

6. Evaluate the expressions:
   a. ln e^4
   b. $ln\left(\frac{1}{e^9}\right)$
7. Use the change of base formula to find the value of $\log_5 100$.
8. What is the difference between $\log_b x^n$ and $(\log_b x)^n$?
9. Condense the expression in order to simplify: 3 log 2 + log 125
10. Is this equation true for any values of x and y? $\log_2 (x+y) = \log_2 x + \log_2 y$
    If so, give the values. If not, explain why not.

Review Answers

1. $\log_b 5 + 2 \log_b x$
2. $4 + 5 \log_3 x$
3. log (x^2 - 1)
4. $ln\left(\frac{x^3y^2}{5x-2}\right)$
5. a. 3
   b. -2
6. a. 4
   b. -9
7. $\frac{log100}{log5} \approx 2.86$
8. The first expression is equivalent to $n \log_b x$. The second expression is the n^{th} power of the log.
9. log 1000 = 3
10. $\log_2 (x+y) = \log_2 x + \log_2 y$ if and only if $x + y = xy$. The solutions to this equation are the possible values of x and y. For example, $x = 3$ and $y = 1.5$

Vocabulary

Common logarithm A common logarithm is a log with base $10k$. The log is usually written without the base.

Natural logarithm A natural log is a log with base e. The natural log is written as ln.

Scientific calculator A scientific calculator is an electronic, handheld calculator that will do calculations beyond the four operations $(+, -, \times, \div)$, such as square roots and logarithms. Graphing calculators will do scientific operations, as well as graphing and equation solving operations.

5.5 Exponential and Logarithmic Models and Equations

Learning objectives

- Analyze data to determine if it can be represented by an exponential or logarithmic model.
- Use a graphing calculator to find an exponential or logarithmic model, and use a model to answer questions about a situation.
- Solve exponential and logarithmic equations using properties of exponents and logarithms.
- Find approximate solutions to equations using a graphing calculator.

Introduction

So far in this chapter we have evaluated exponential and logarithm expressions, and we have graphed exponential and logarithmic functions. In this lesson you will extend what you have learned in two ways. First, we will introduce the idea of modeling real phenomena with logarithmic and exponential functions. Second, we will solve logarithmic and exponential equations. While you have already solved some equations in previous lessons, now you will be able to solve more complicated equations. This lesson will provide you with further tools for the applications of logarithmic and exponential functions that will be the focus of the remainder of the chapter.

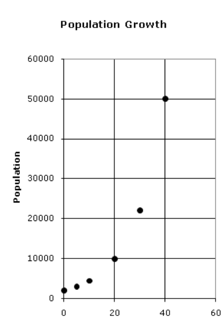
Exponential Models

Consider the following example: the population of a small town was 2,000 in the year 1950. The population increased over time, as shown by the values in the table:

TABLE 5.34:

Year (1950 = 0)	Population
0	2000
5	2980
10	4450
20	9900
30	22,000
40	50,000

If you plot these data points, you will see that the growth pattern is non-linear:

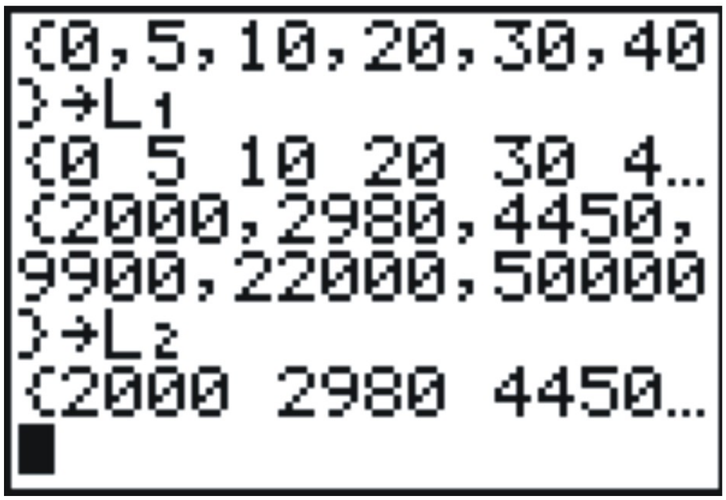


In many situations, population growth can be modeled with an exponential function (this is because population grows as a percentage of the current population, i.e. 8% per year). In lesson 7, you will learn how to create such models using information from a given situation. Here, we will focus on creating models using data and a graphing calculator.

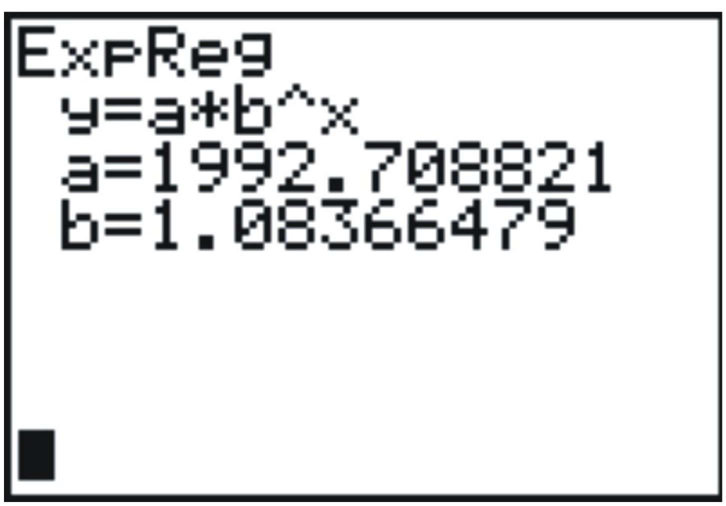
The population data from the example above can be modeled with an exponential function, but the function is not unique. That is, there is more than one way to write a function to model this data. In the steps below you will see how to use a graphing calculator to find a function of the form $y = a(b^x)$ that fits the data in the table.

Technology Note *Using a TI-83/84 graphing calculator to find an exponential function that best fits a set of data* 1. Entering the dataa. Data must be entered into lists. The calculator has six named lists, L1, L2, ... L6. We will enter the *x* values in L1 and the *y* values in L2. One way to do this is shown below:

Press <TI font_2nd>[{] and then enter the numbers separated by commas, and close by pressing the following: <TI font_2nd>[{]<TI font_STO><TI font_2nd>[L1]. The top three lines of the figure below show the entry into list L1, followed by the entry of the *y* values into list L2.



Now press <TI font_STAT>, and move to the right to the CALC menu. Scroll down to option 10, **ExpReg.** Press <TI font_ENTER>, and you will return to the home screen. You should see **ExpReg** on the screen. As long as the numbers are in L1 and L2, the calculator will proceed to find an exponential function to fit the data you listed in List L1 and List L2. You should see on the home screen the values for *a* and *b* in the exponential function (See figure below). Therefore the function $y = 1992.7(1.0837)^x$ is an approximate model for the data.

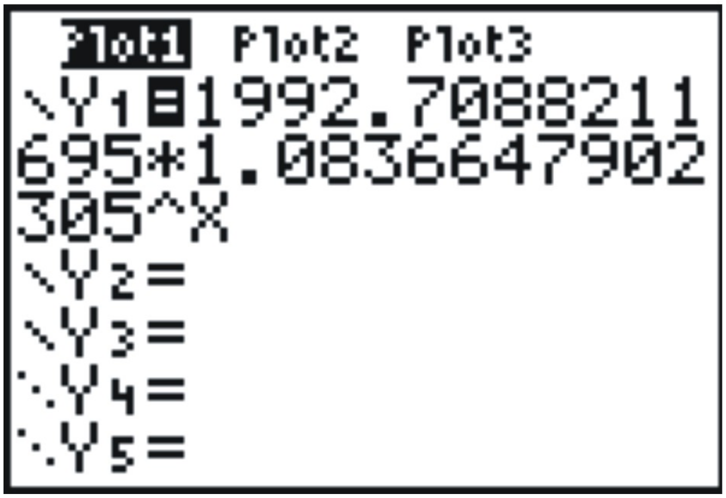


2. **Plotting the data and the equation** To view plots of the data points and the equation on the same screen, do the following.

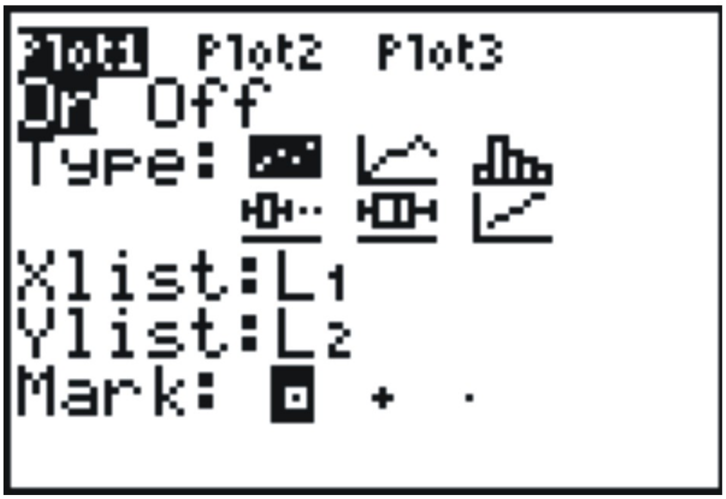
a. First, press <TI font_Y=>and clear any equations.

You can type in the equation above, or to get the equation from the calculator, do the following:

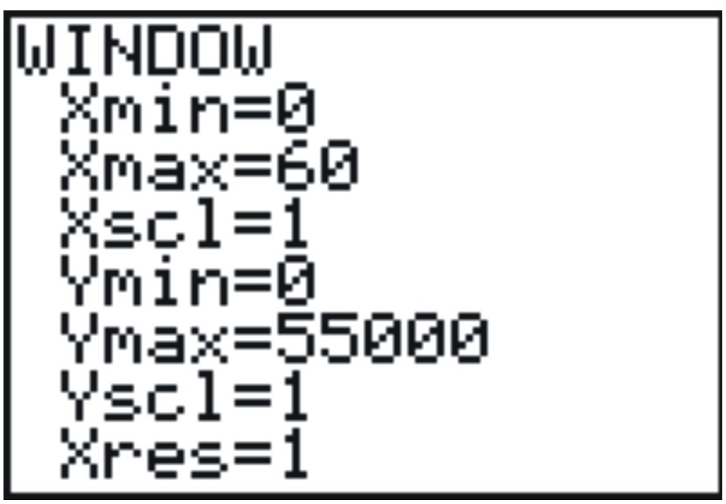
b. Enter the above rounded-off equation in Y1, or use the following procedure to get the full equation from the calculator: put the cursor in Y1, press <TI font_VARS>, 5, EQ, and 1. This should place the equation in Y1 (see figure below).



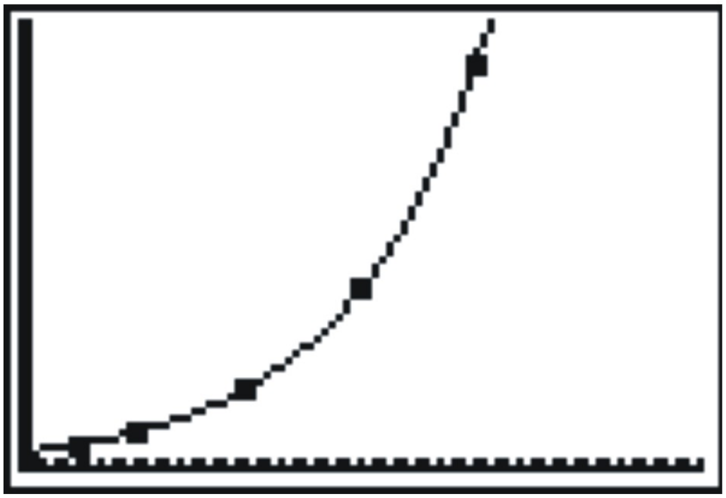
c. Now press <TI font_2nd>[STAT PLOT] and complete the items as shown in the figure below.



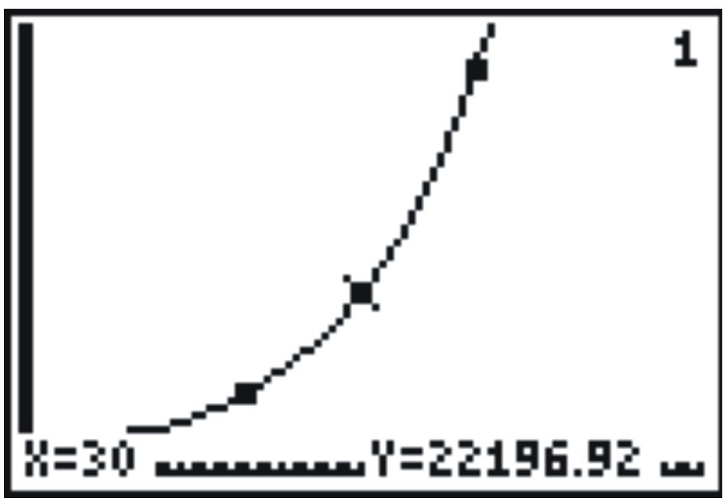
d. Now set your window. (Hint: use the range of the data to choose the window the figure below shows our choices.)



e. Press <TI font GRAPH>and you will to see the function and the data points as shown in the figure below.



3. Comparing the real data with the modeled results It looks as if the data points lie on the function. However, using the TRACE function you can determine how close the modeled points are to the real data. Press <TI font_TRACE>to enter the TRACE mode. Then press the right arrow to move from one data point to another. Do this until you land on the point with value Y=22000. To see the corresponding modeled value, press the up or down arrow. See the figure below. The modeled value is approximately 22197, which is quite close to the actual data. You can verify any of the other data points using the same method.



Now that we have the equation $y = 1992.7(1.0837)^x$ to model the situation, we can estimate the population for any years that were not in the original data set. If we choose x values between 0 and 40, it is called **interpolation**. If we choose other x values outside of this domain, it is called **extrapolation**. Interpolation is, in a sense, a safer way of estimating population, because it is within the data points that we have, and does not require that we think about the end behavior of the function. For example, if we extrapolate to the year 1930, this means x = -20. The function value is 399. However, if the town was founded in 1940, then this data value does not make any sense. Similarly, if we extrapolate to the year 2000, we have x = 50. The function value is 110,711. However, if the towns pattern of population growth shifted (perhaps due to some economic change), this estimation could be highly inaccurate. As noted above, you will study exponential growth, as well as other exponential models, in the next two lessons. Now we turn to logarithmic models.

Logarithmic Models

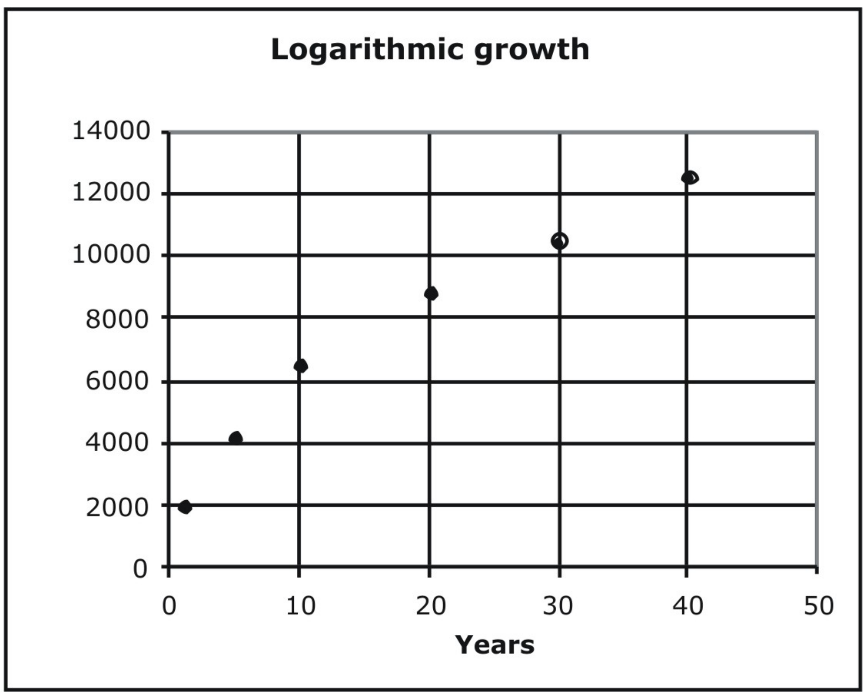
Consider another example of population growth:

Table 2

TABLE 5.35:

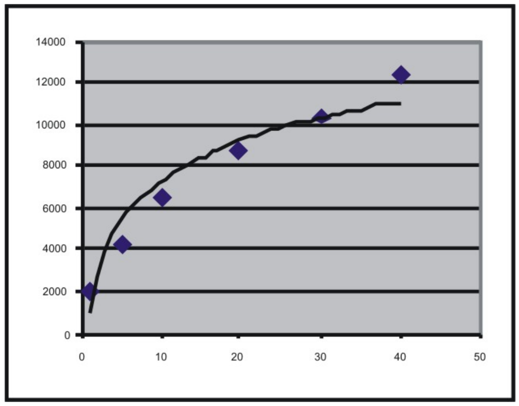
Year	Population
1	2000
5	4200
10	6500
20	8800
30	10500
40	12500

If we plot this data, we see that the growth is not quite linear, and it is not exponential either.



Just as we found an exponential model in the previous example, here we can find a logarithmic function to model this data. First enter the data in the table in L1 and L2. Then press STAT to get to the CALC menu. This time choose option 9. You should get the function $y = 930.4954615 + 2780.218173 \ln x$. If you view the graph and the data points together, as described in the Technology Note above, you will see that the graph of the function does not touch the data points, but models the general trend of the data.

Note about technology: you can also do this using an Excel spreadsheet. Enter the data in a worksheet, and create a scatterplot by inserting a chart. After you create the chart, from the chart menu, choose add trendline. You will then be able to choose the type of function. Note that if you want to use a logarithmic function, the domain of your data set must be positive numbers. The chart menu will actually not allow you to choose a logarithmic trendline if your data include zero or negative x values. See below:



Solve Exponential Equations

Given an exponential model of some phenomena, such as population growth, you may want to determine a particular input value that would produce a given function value. Lets say that a function $P(x) = 2000(1.05)^x$ models the population growth for a town. What if we want to know when the population reaches 20,000?

To answer this equation, we must solve the equation $2000(1.05)^x = 20{,}000$. We can solve this equation by isolating the power $(1.05)^x$ and then using one of the log properties:

TABLE 5.36:

$2000(1.05)^x = 20,000$	Divide both sides of the equation by 2000
$(1.05)^x = 10$	Take the common log of both sides
$log(1.05)^x = log10$	Use the power property of logs
$xlog(1.05) = log10$	Evaluate $log10$
$xlog(1.05) = 1$	
$x = \frac{1}{log(1.05)} \approx 47$	Divide both sides by $log(1.05)$
	Use a calculator to estimate $log(1.05)$

We can use these same techniques to solve any exponential equation.

Example 1: Solve each exponential equation

TABLE 5.37:

a. $2^x + 7 = 19$	b. $3^{5x-1} = 16$

Solution:

a. $2^x + 7 = 19$

$-7 - 7$

$2^x = 12$

$log2^x = log12$

$xlog2 = log12$

$x = \frac{log12}{log2} \approx 3.58$

b. $3^{5x-1} = 16$

$log3^{5x-1} = log16$

$(5x-1)log3 = log16$

$5x-1 = \frac{log16}{log3}$

$5x = \frac{log16}{log3}+1$

$x = \frac{\frac{log16}{log3}+1}{5}$

$x \approx 0.705$

Solve Logarithmic Equations

In the previous lesson we solved two forms of log equations. Now we can solve more complicated equations, using our knowledge of log properties. For example, consider the equation $\log_2(x) + \log_2(x - 2) = 3$. We can solve this equation using a log property.

TABLE 5.38:

$\log_2(x) + \log_2(x - 2) = 3$	
$\log_2(x(x - 2)) = 3$	$\log_b x + \log_b y = \log_b(xy)$
$\log_2(x^2 - 2x) = 3 \Rightarrow$	write the equation in exponential form.
$2^3 = x^2 - 2x$	
$x^2 - 2x - 8 = 0$	Solve the resulting quadratic
$(x - 4)(x + 2) = 0$	
$x = -2, 4$	

The resulting quadratic has two solutions. However, only $x = 4$ is a solution to our original equataion, as $\log_2(-2)$ is undefined. We refer to $x = -2$ as an **extraneous solution.**

Example 2: Solve each equation

TABLE 5.39:

a. $\log(x + 2) + \log 3 = 2$	b. $\ln(x + 2) - \ln(x) = 1$

Solution:

a. $\log(x + 2) + \log 3 = 2$

TABLE 5.40:

$\log(3(x + 2)) = 2$	$\log_b x + \log_b y = \log_b(xy)$
$\log(3x + 6) = 2$	Simplify the expression $3(x+2)$
$10^2 = 3x + 6$	Write the log expression in exponential form
$100 = 3x + 6$	
$3x = 94$	Solve the linear equation
$x = 94/3$	

b. $\ln(x + 2) - \ln(x) = 1$

TABLE 5.41:

$ln\left(\frac{x+2}{x}\right)=1$	$log_b x - log_b y = log_b\left(\frac{x}{y}\right)$
$e^1=\frac{x+2}{x}$	Write the log expression in exponential form.
$ex=x+2$	Multiply both sides by x.
$ex-x=2$	Factor out x.
$x(e-1)=2$	Isolate x.
$x=\frac{2}{e-1}$	

The solution above is an *exact* solution. If we want a decimal approximation, we can use a calculator to find that x 1.16. We can also use a graphing calculator to find an approximate solution, as we did in lesson 2 with exponential equations. Consider again the equation ln (x + 2) - ln (x) = 1. We can solve this equation by solving a system:

$$\begin{cases} y = ln(x+2) - ln(x) \\ y = 1 \end{cases}$$

If you graph the system on your graphing calculator, as we did in lesson 2, you should see that the curve and the horizontal line intersection at one point. Using the **INTERSECT** function on the **CALC** menu (press <TI font_2nd>[CALC]), you should find that the x coordinate of the intersection point is approximately 1.16. This method will allow you to find approximate solutions for more complicated log equations.

Example 3. Use a graphing calculator to solve each equation:

TABLE 5.42:

a. log(5 - x) + 1 = log x	b. $\log_2(3x + 8) + 1 = \log_3(10 - x)$

Solution:

a. log(5 - x) + 1 = log x

The graphs of y = log (5 - x) + 1 and y = log x intersect at x 4.5454545

Therefore the solution of the equation is x 4.54.

b. $\log_2(3x + 8) + 1 = \log_3(10 - x)$

First, in order to graph the equations, you must rewrite them in terms of a common log or a natural log. The resulting equations are: $y=\frac{log(3x+8)}{log2}+1$ and $y=\frac{log(10-x)}{log3}$. The graphs of these functions intersect at x -1.87. This value is the approximate solution to the equation.

Lesson Summary

This lesson has introduced the idea of modeling a situation using an exponential or logarithmic function. When a population or other quantity has a steep increase over time, it may be modeled with an exponential function. When a population has a steep increase, but then slower growth, it may be modeled with a logarithmic function. (In a later lesson you will learn about a third option.) We have also examined techniques of solving exponential and logarithmic equations, based on our knowledge of properties of logarithms. The key property to remember is the power property:

$$\log_b x^n = n \log_b x$$

Using this property allows us to turn an exponential function into a linear function, which we can then solve in order to solve the original exponential function.

In the remaining lessons in this chapter, you will learn about several different real phenomena that are modeled with exponential and logarithmic equations. In these lessons you will also use the techniques of equation solving learned here in order to answer questions about these phenomena.

Points to Consider

1. What kinds of situations might be modeled with exponential functions or logarithmic functions?
2. What restrictions are there on the domain and range of data if we use these functions as models?
3. When might an exponential or logarithmic equation have no solution?
4. What are the advantages and disadvantages of using a graphing calculator to solve exponential and logarithmic equations?

Review Questions

For questions 1 - 5, solve each equation using algebraic methods. Give an exact solution.

1. $2\ (5^{x\ -4}) + 7 = 43$
2. $4^x = 7^{3x\ -5}$
3. $\log(5x + 200) + \log 2 = 3$
4. $\log_3\ (4x + 5)$ - $\log_3 x = 2$
5. $\ln\ (4x + 1)$ - $\ln\ (2x) = 3$
6. Use a graphing utility to solve the equation in #4.
7. Use a graphing utility to solve the equation $\log(x^2$ -3) = $\log\ (x + 5)$
8. In example 3b, the solution to the log equation $\log_2(3x + 8) + 1 = \log_3\ (10$ - $x)$ was found to be x - 1.87. One student read this example, and wondered how the value of x could be negative, given that you cannot take a log of a negative number. How would you explain to this student why the solution is valid?
9. The data set below represents a hypothetical situation: You invest \$2000 in a money market account, and you do not invest more money or withdraw any from the account.
   Table 3

TABLE 5.43:

Time since you invested (in years)	Amount in account
0	2,000
2	2200
5	2500
10	3300
20	4500

10.
11. a. Use a graphing utility to find an exponential model for the data. b. Use your model to estimate the value of the account after the 8
12. th
13. year. c. At this rate, much money would be in the account after 30 years? d. Explain how your estimate in part c might be inaccurate. (What might happen after 20 years?)
14. The data set below represents the growth of a plant.
    Table 4

TABLE 5.44:

Time since planting (days)	Height of plant (inches)
1	.2
4	.5
5	.57
10	1.2
12	1.3

TABLE 5.44: (continued)

Time since planting (days)	Height of plant (inches)
14	1.4

15.
16. a. Use a graphing utility to find a logarithmic equation to model the data. b. Use your model to estimate the height of the plant after 15 days. Compare this estimate to the trend in the data. c. Give an example of an
17. *x*
18. value for which the model does not make sense.
19. In the lesson, the equation log(5 - *x*) + 1 = log *x* was solved using a graph. Solve this equation algebraically in order to (a) verify the approximate solution found in the lesson and (b) give an exact solution.

Review Answers

1. $log_5 18+4$ or $\frac{log18}{log5}+4$
2. $\frac{-5log7}{log4-3log7}$
3. x = 60
4. x = 1
5. $x=\frac{1}{2e^3-4}$
6. The function $y = \log_3 (4x + 5) - \log_3 x$ intersects the line $y = 2$ at the point (1, 2)
7. The graphs intersect twice, giving 2 solutions: *x* -2.37, *x* 3.37
8. The value of can be negative as long as the argument of the log is positive. In this equation, the arguments are 3*x*+8 and 10-*x*. Neither expression takes on a negative value for x -1.87
9. a. $y = 2045.405(1.042)^x$
   b. About $2840
   c. About $7003
   d. After that much time, you may decide to withdraw the money to spend or to invest in something with more potential for growth.
10. a. $y = 0.0313 + .4780 \ln x$
    b. The model gives 1.32 inches. The data would suggest the plant is at least 1.4 inches tall.
    c. The model does not make sense for negative *x* values. Also, at some point the plant could die. This reality puts an upper bound on *x*.
11. log (5 - *x*) + 1 = log *x*
    log (5 - *x*) - log *x* + 1 = 0
    log(5 - *x*) - log *x* = -1
    $log\left(\frac{5-x}{x}\right)=-1$
    $10^{-1}=\frac{5-x}{x}$
    $0.1=\frac{5-x}{x}$
    0.1 *x* = 5 - *x*
    1.1 *x* = 5
    $x=\frac{5}{1.1}=\frac{50}{11}=4\frac{6}{11}=5.\overline{54}$

Vocabulary

Extraneous solution An extraneous solution is a solution to an equation used to solve an initial equation that is not a solution to the initial equation. Extraneous solutions occur when solving certain kinds of equations, such as log equations, or square root equations.

Extrapolation To extrapolate from data is to create new data points, or to predict, outside of the domain of the data set.

Interpolation To interpolate is to create new data points, or to predict, within the domain of the data set, but for points not in the original data set.

5.6 Compound Interest

Learning objectives

- Calculate compound interest, including continuous compounding.
- Compare compound interest situations.
- Determine algebraically and graphically the time it takes for an account to reach a particular value.

Introduction

In the previous lesson, you learned about modeling growth using an exponential function. In this lesson we will focus on a specific example of exponential growth: compounding of interest. We will begin with the case of simple interest, which refers to interest that is based only on the principal, or initial amount of an investment or loan. Then we will look at what it means for interest to compound. In the simplest terms, compounding means that interest accrues (you gain interest on an investment, or owe more on a loan) based on the principal you invested, as well as on interest you have already accrued. As you will soon see, compound interest is a case of exponential growth. In this lesson we will look at specific examples of compound interest, and we will write equations to model these specific situations.

Simple interest over time

As noted above, simple interest means that interest accrues based on the principal of an investment or loan. The simple interest is calculated as a percent of the principal. The formula for simple interest is, in fact, simple:

The variable P represents the principal amount, r represents the interest rate, and t represents the amount of time the interest has been accruing. For example, say you borrow $2,000 from a family member, and you insist on repaying with interest. You agree to pay 5% interest, and to pay the money back in 3 years. The interest you will owe will be 2000(0.05)(3) = $300. This means that when you repay your loan, you will pay $2300. Note that the interest you pay after 3 years is not 5% of the original loan, but 15%, as you paid 5% of $2000 each year for 3 years.

Now lets consider an example in which interest is compounded. Say that you invest $2000 in a bank account, and it earns 5% interest annually. How much is in the account after 3 years?

In order to determine how much money is in the account after three years, we have to determine the amount of money in the account after each year. The table below shows the calculations for one, two, and three years of this investment:

TABLE 5.45:

Year	Principal + interest
After one year	2000 + 2000(0.05) = 2000 + 100 = $2100
After 2 years	2100 + 2100(0.05) = 2100 + 105 = $2205
After 3 years	2205 + 2205(0.05) = 2205 + 110.25 = $2315.25

Therefore, after three years, you will have $2315.25 in the account, which means that you will have earned $315.25 in interest. With simple interest, you would have earned $300 in interest. Compounding results in more interest because the principal on which the interest is calculated increased each year. For example, in the second year shown in the table above, you earned 5% of 2100, *not* 5% of 2000, as would be the case of simple interest. The main idea here is that compounding creates more interest because you are earning interest on interest, and not just on the principal.

But how much more?

You might look at the above example and say, its only $15.25. Remember that we have only looked at one example, and this example is a small one: in the grand scheme of investing, $2000 is a small amount of money, and we have only looked at the growth of the investment for a short period of time. For example, if you are saving for retirement, you could invest for a period of 30 years or more, and you might invest several thousand dollars each year.

The formulas and methods for calculating retirement investments are more complicated than what we will do here. However, we can use the above example to derive a formula that will allow us to calculate compound interest for any number of years.

The compound interest formula

To derive the formula for compound interest, we need to look at a more general example. Lets return to the previous example, but instead of assuming the investment is $2000, let the principal of the investment be P dollars. The key idea is that each year you have 100% of the principal, plus 5% of the previous balance. The table below shows the calculations of this more general investment.

TABLE 5.46:

Year	Principal + interest	New principal
1	P + P .05 = 1.00P + .05P =	1.05P
2	1.05P + .05 (1.05P) = 1.05P [1+ .05] = 1.05P 1.05 =	$(1.05)^2$ P
3	$(1.05)^2$ P + .05 $(1.05)^2$ P = $(1.05)^2$ P [1 + .05] = $(1.05)^2$ P [1.05] =	$(1.05)^3$ P

Notice that at the end of every year, the amount of money in the investment is a power of 1.05, times P, and that the power corresponds to the number of years. Given this pattern, you might hypothesize that after 4 years, the amount of money is $(1.05)^4P$.

We can generalize this pattern to a formula. As above, we let P represent the principal of the investment. Now, let t represent the number of years, and r represent the interest rate. Keep in mind that 1.05 = 1+0.05. So we can generalize:

$$A(t) = P(1 + r)^t$$

This function will allow us to calculate the amount of money in an investment, if the interest is compounded each year for t years.

Example 1: Use the formula above to determine the amount of money in an investment after 20 years, if you invest $2000, and the interest rate is 5% compounded annually.

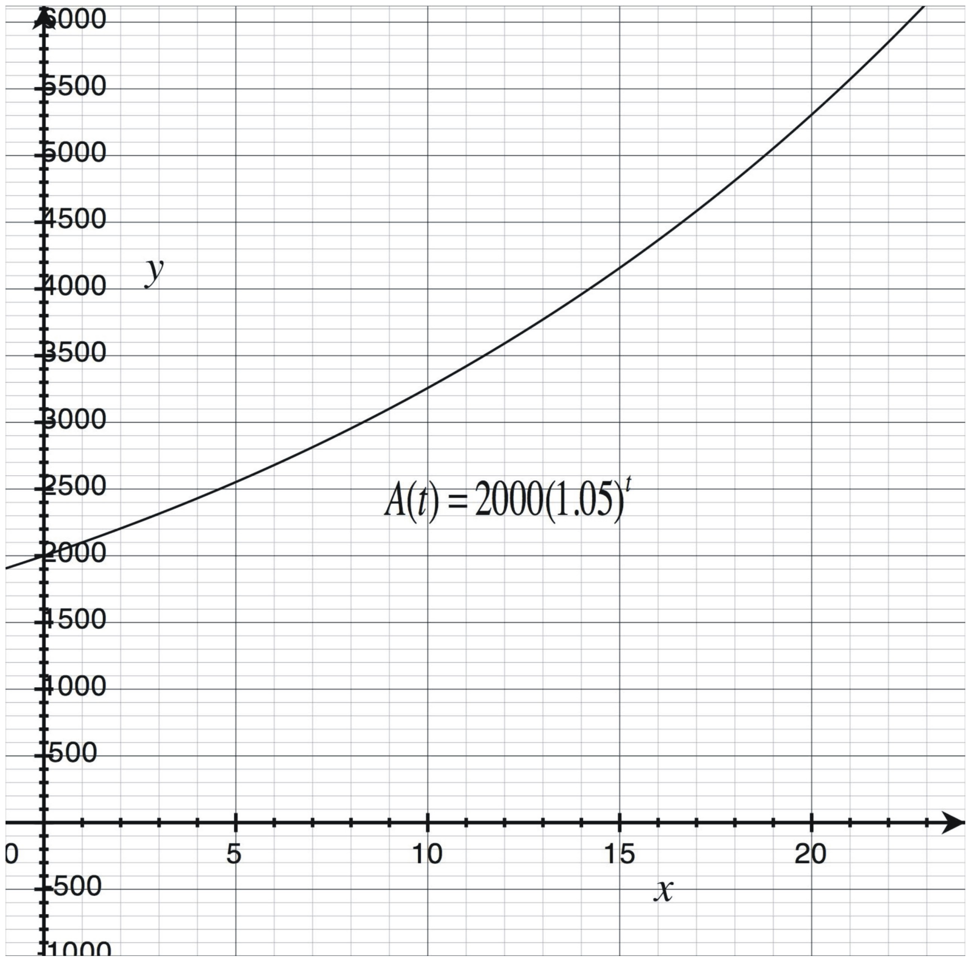
Solution: The investment will be worth $5306.60

$A(t) = P(1 + r)^t$

$A(20) = 2000(1.05)^{20}$

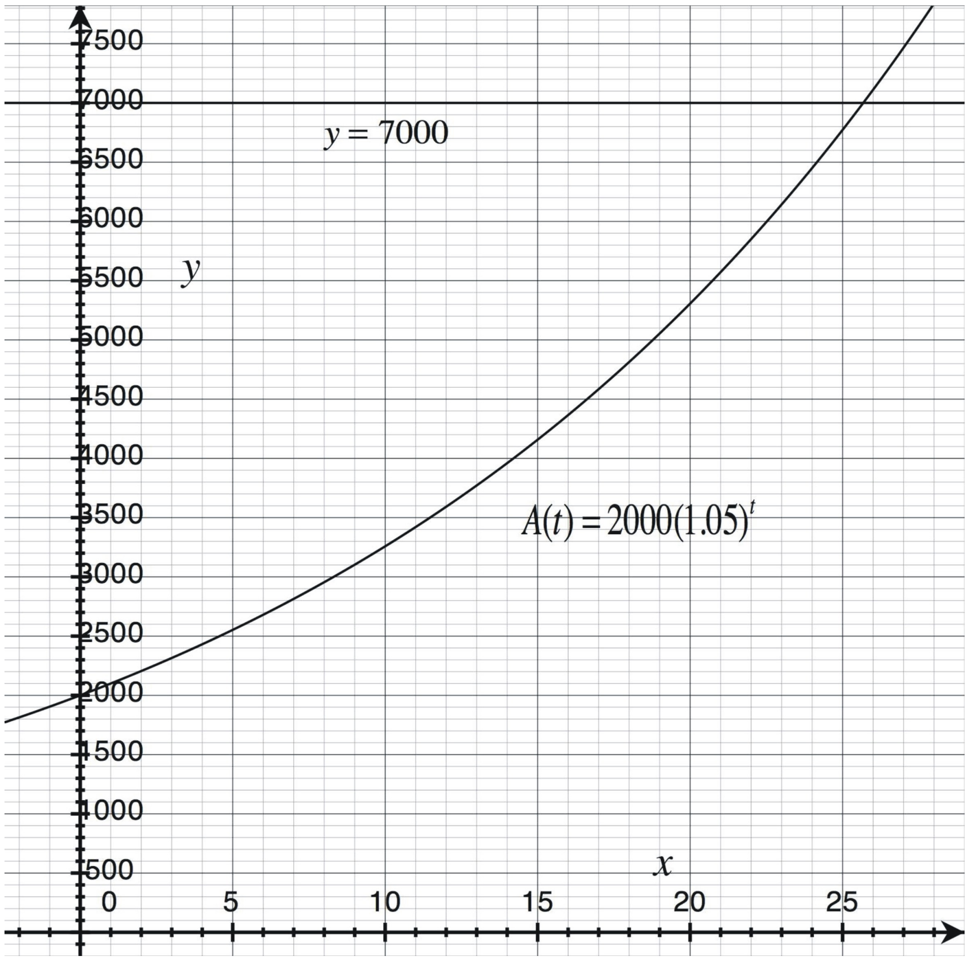
$A(20) = \$5306.60$

In the above example, we found the value of this investment after a particular number of years. If we graph the function $A(t) = 2000(1.05)^t$, we can see the values for any number of years.



If you graph this function using a graphing calculator, you can determine the value of the investment by tracing along the function, or by pressing <TI font_TRACE>on your graphing calculator and then entering an x value. You can also choose an investment value you would like to reach, and then determine the number of years it would take to reach that amount. For example, how long will it take for the investment to reach $7,000?

As we did earlier in the chapter, we can find the intersection of the exponential function with the line $y = 7000$.



You can see from Figure 2 that the line and the curve intersect at a little less than $x = 26$. Therefore it would take almost 26 years for the investment to reach $7000.

You can also solve for an exact value:

TABLE 5.47:

$2000(1.05)^t = 7000$	
$(1.05)^t = \frac{7000}{2000}$	Divide both sides by 2000 and simplify the right side of the equation.
$(1.05)^t = 3.5$	Take the ln of both sides (you can use any log, but ln or log base 10 will allow you to use a calculator.)
$ln(1.05)^t = ln3.5$	
$t[ln(1.05)] = ln3.5$	Use the power property of logs
$t = \frac{ln3.5}{ln1.05} \approx 25.68$	Divide both sides by ln 1.05

The examples we have seen so far are examples of annual compounding. In reality, interest is often compounded more frequently, for example, on a monthly basis. In this case, the interest rate is divided amongst the 12 months. The formula for calculating the balance of the account is then slightly different:

$$A(t) = P\left(1 + \frac{r}{12}\right)^{12t}$$

Notice that the interest rate is divided by 12 because $1/12^{th}$ of the rate is applied each month. The variable t in the exponent is multiplied by 12 because the interest is calculated 12 times per year.

In general, if interest is compounded n times per year, the formula is:

$A(t) = P\left(1+\frac{r}{n}\right)^{nt}$

Example 2: Determine the value of each investment.

a. You invest \$5000 in an account that gives 6% interest, compounded monthly. How much money do you have after 10 years? b. You invest \$10,000 in an account that gives 2.5% interest, compounded quarterly. How much money do you have after 10 years?

Solution:

a. \$5000, invested for 10 years at 6% interest, compounded monthly.$A(t) = P\left(1+\frac{r}{n}\right)^{nt}$ $A(10) = 5000\left(1+\frac{.06}{12}\right)^{12\cdot 10}$ $A(10) = 5000\left(1.005\right)^{120}$ $A(10) \approx \$9096.98$

b. \$10000, invested for 10 years at 2.5% interest, compounded quarterly.

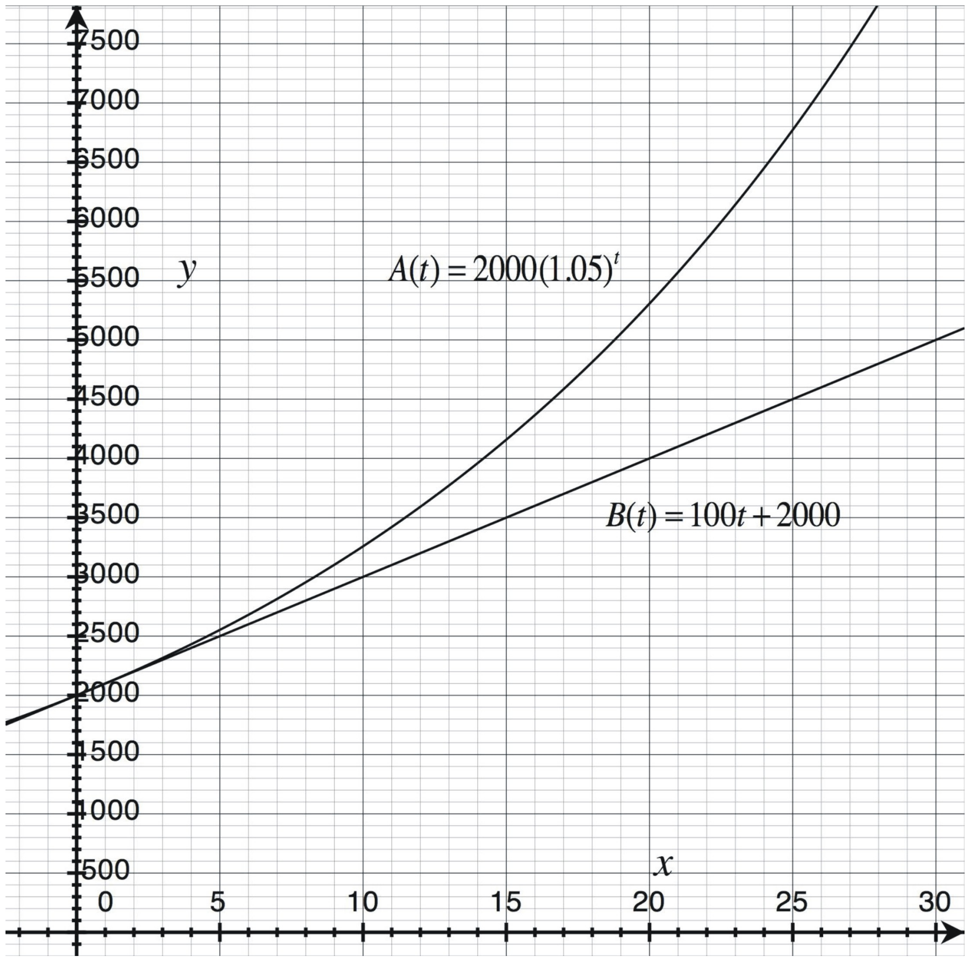
Quarterly compounding means that interest is compounded four times per year. So in the equation, n = 4.

$A(t) = P\left(1+\frac{r}{n}\right)^{nt}$ $A(10) = 6000\left(1+\frac{.025}{4}\right)^{4\cdot 10}$ $A(10) = 6000(1.00625)^{40}$ $A(10) \approx \$12,830.30$

In each example, the value of the investment after 10 years depends on three quantities: the principal of the investment, the number of compoundings per year, and the interest rate. Next we will look at an example of one investment, but we will vary each of these quantities.

The power of compound interest

Consider the investment in example 1: \$2000 was invested at an annual interest rate of 5%. We modeled this situation with the equation $A(t) = 2000(1.05)^t$. We can use this equation to determine the amount of money in the account after any number of years. As we saw above, the value of the account grows exponentially. You can see how fast the investment grows if we compare it to linear growth. For example, if the same investment earned simple interest, the value of the investment after t years could be modeled with the function $B(t) = 2000 + 2000(.05)t$. We can simplify this to be: $B(t) = 100t + 2000$. The exponential function and this linear function are shown here.



Notice that if we look at these investments over a long period of time (30 years are shown in the graph), the values look very close together for x values less than 10. For example, after 5 years, the compound interest investment is worth \$2552.60, and the simple interest investment is worth \$2500. But, after 20 years, the compound interest investment is worth \$5306.60, and the simple interest investment is worth \$4,000. After 20 years, simple interest has doubled the amount of money, while compound interest has resulted in 2.65 times the amount of money.

The main idea here is that an exponential function grows faster than a linear one, which you can see from the graphs above. But what happens to the investment if we change the interest rate, or the number of times we compound per year?

Example 3: Compare the values of the investments shown in the table. If everything else is held constant, how does the interest rate influence the value of the investment?

TABLE 5.48:

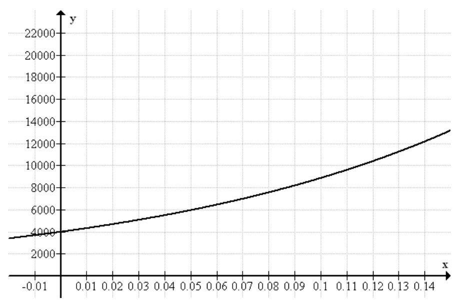
	Principal	r	n	t
a.	\$4,000	.02	12	8
b.	\$4,000	.05	12	8
c.	\$4,000	.10	12	8
d.	\$4,000	.15	12	8
e.	\$4,000	.22	12	8

Solution: Using the compound interest formula $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$, we can calculate the value of each investment. In all cases, we have $A(8) = 4000\left(1+\frac{r}{12}\right)^{12\cdot 8}$.

TABLE 5.49:

	Principal	r	n	t	A
a.	\$4,000	.02	12	8	\$4693.42
b.	\$4,000	.05	12	8	\$5962.34
c.	\$4,000	.10	12	8	\$8872.70
	\$4,000	.15	12	8	\$13182.05
e.	\$4,000	.22	12	8	\$22882.11

As we increase the interest rate, the value of the investment increases. It is part of every day life to want to find the highest interest rate possible for a bank account (and the lowest possible rate for a loan!). Lets look at just how fast the value of the account grows. Remember that each calculation in the table above started with $A(8) = 4000\left(1+\frac{r}{12}\right)^{12\cdot 8}$. Notice that this is a function of r, the interest rate. We can write this equation in a more standard form: $f(x) = 4000\left(1+\frac{x}{12}\right)^{96}$. The graph of this function is shown below:



Notice that while this function is not exponential, it does grow quite fast. As we increase the interest rate, the value of the account increases very quickly.

Example 4: Compare the values of the investments shown in the table. If everything else is held constant, how does the compounding influence the value of the investment?

TABLE 5.50:

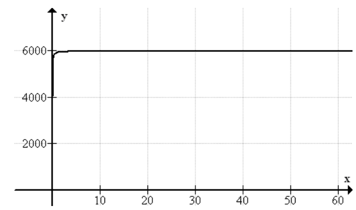
	Principal	r	n	t
a.	\$4,000	.05	1 (annual)	8
b.	\$4,000	.05	4 (quarterly)	8
c.	\$4,000	.05	12 (monthly)	8
d.	\$4,000	.05	365 (daily)	8
e.	\$4,000	.05	8760 (hourly)	8

Solution: Again, we use the compound interest formula. For this example, the n is the quantity that changes: $A(8) = 4000\left(1+\frac{.05}{n}\right)^{8n}$

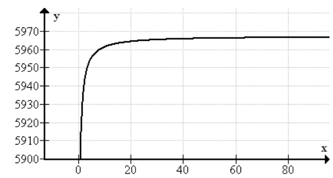
TABLE 5.51:

	Principal	r	n	t	A
a.	\$4,000	.05	1 (annual)	8	\$5909.82
b.	\$4,000	.05	4 (quarterly)	8	\$5952.52
c.	\$4,000	.05	12 (monthly)	8	\$5962.34
d.	\$4,000	.05	365 (daily)	8	\$5967.14
e.	\$4,000	.05	8760 (hourly)	8	\$5967.29

In contrast to the changing interest rate, in this example, increasing the number of compoundings per year does not seem to dramatically increase the value of the investment. We can see why this is the case of we look at the function $A(8) = 4000\left(1+\frac{.05}{n}\right)^{8n}$. A graph of the function $f(x) = 4000\left(1+\frac{.05}{x}\right)^{8x}$ is shown below:



The graph seems to indicate that the function has a horizontal asymptote at $6000. However, if we zoom in, we can see that the horizontal asymptote is closer to 5967.



What does this mean? This means that for the investment of $4000, at 5% interest, for 8 years, compounding more and more frequently will never result in more than about $5968.00.

Another way to say this is that the function $f(x) = 4000\left(1+\frac{.05}{x}\right)^{8x}$ has a limit as x approaches infinity. Next we will look at this kind of limit to define a special form of compounding.

Continuous compounding

Consider a hypothetical example: you invest $1.00, at 100% interest, for 1 year. For this situation, the amount of money you have at the end of the year depends on how often the interest is compounded:

$$A(t) = P\left(1+\frac{r}{n}\right)^{nt}$$

$$A(t) = 1\left(1+\frac{1}{n}\right)^{1n}$$

$$A = \left(1+\frac{1}{n}\right)^{n}$$

Now lets consider different compoundings:

TABLE 5.52:

Compounding	N	A ≈
Annual	1	2
Quarterly	4	2.44140625
Monthly	12	2.61303529022
Daily	365	2.71456748202
Hourly	8,760	2.71812669063
By the minute	525,600	2.7182792154
By the second	31,536,000	2.71828247254

The values of A in the table have a limit, which might look familiar: its the number e. In fact, one of the definitions of e is $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

A related limit is one that will lead us to a special kind of compound interest: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$. (The proof of this limit requires calculus. However, in one of the review questions, you will examine this limit more closely.)

Now we can define what is known as continuous compounding. If interest is compounded n times per year, the equation we use is: $A(t) = P\left(1+\frac{r}{n}\right)^{nt}$. We can also write the function as $A(t) = P\left(\left(1+\frac{r}{n}\right)^n\right)^t$. If we compound more and more often, we are looking at what happens to this function as $n \to \infty$. Recall the limit above: $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$. Here, this means $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$. So as n approaches ∞, $\left(\left(1+\frac{r}{n}\right)^n\right)^t$ approaches $P(e^r)^t = Pe^{rt}$.

The function $A(t) = Pe^{rt}$ is the formula we use to calculate the amount of money when interest is continuously compounded, rather than interest that is compounded at discrete intervals, such as monthly or quarterly. For example, consider again the investment in example 1 above: what is the value of an investment after 20 years, if you invest \$2000, and the interest rate is 5% compounded continuously?

$A(t) = Pe^{rt}$ $A(20) = 2000e^{.05(20)}$ $A(20) = 2000e^1$ $A(20) = \$5436.56$

Just as we did with the standard compound interest formula, we can also determine the time it takes for an account to reach a particular value if the interest is compounded continuously.

Example 5: How long will it take \$2000 to grow to \$25,000 in the previous example?

Solution: It will take about 50 years:

TABLE 5.53:

$A(t) = Pe^{rt}$	
$25{,}000 = 2000e^{.05(t)}$	
$12.5 = e^{.05(t)}$	Divide both sides by 2000
$\ln 12.5 = \ln e^{.05(t)}$	Take the ln of both sides
$\ln 12.5 = .05t \ln e$	Use the power property of logs
$\ln 12.5 = .05t\ 1$	lne = 1
$\ln 12.5 = 0.5t$	Isolate t
$t = \frac{ln 12.5}{.05} \approx 50.5$	

Lesson Summary

In this lesson we have developed formulas to calculate the amount of money in a bank account or an investment when interest is compounded, either a discrete number of times per year, or compounded continuously. We have found the value of accounts or investments, and we have found the time it takes to reach a particular value. We have solved these problems algebraically and graphically, using our knowledge of functions in general, and logarithms in particular.

In general, the examples we have seen are conservative in the larger scheme of investing. Given all of the information available today about investments, you may look at the examples and think that the return on these investments seems low. For example, in the last example, 50 years probably seems like a long time to wait!

It is important to keep in mind that these calculations are based on an initial investment only. In reality, if you invest money long term, you will invest on a regular basis. For example, if an employer offers a retirement plan, you may invest a set amount of money from every paycheck, and your employer may contribute a set amount as well. As noted above, the calculations for the growth of a retirement investment are more complicated. However, the exponential functions you have studied in this lesson are the basis for the calculations you would need to do. The examples here are meant to illustrate an application of exponential functions, and the power of compound interest.

Points to Consider

1. Why is compound interest modeled with an exponential function?
2. What is the difference between compounding and continuous compounding?
3. How are logarithms useful in solving compound interest problems?

Review Questions

1. You put \$3500 in a bank account that earns 5.5% interest, compounded monthly. How much is in your account after 2 years? After 5 years?
2. You put \$2000 in a bank account that earns 7% interest, compounded quarterly. How much is in your account after 10 years?
3. Solve an exponential equation in order to answer the question: given the investment in question 2, how many years will it take for the account to reach \$10,000?
4. Use a graph to verify your answer to question 3.
5. Consider two investments:
   (1) \$2000, invested at 6% interest, compounded monthly
   (2) \$3000, invested at 4.5% interest, compounded monthly
   Use a graph to determine when the 2 investments have equal value.
6. You invest \$3000 in an account that pays 6% interest, compounded monthly. How long does it take to double your investment?
7. Explain why the answer to #6 does not depend on the amount of the initial investment
8. You invest \$4,000 in an account that pays 3.2% interest, compounded continuously. What is the value of the account after 5 years?
9. You invest \$6,000 in an account that pays 5% interest, compounded continuously. What is the value of the account after 10 years?
10. Consider the investment in example 8. How many years will it take the investment to reach \$20,000?
11. In this lesson, we introduced this limit: $\lim n \to \infty \left(1+\frac{r}{n}\right)^n = e^r$
    We noted in the lesson that the proof of this limit requires calculus. Here, we will examine a few specific cases in order to see how this limit is true.
    Using a graphing calculator, estimate the value of this limit for the given values of x. Do the limits seem to match the value of e^x*? Do these calculations suffice as a proof of the limit? Why or why not?*
    Hint: Graph each limit expression as a function, where x represents n.

TABLE 5.54:

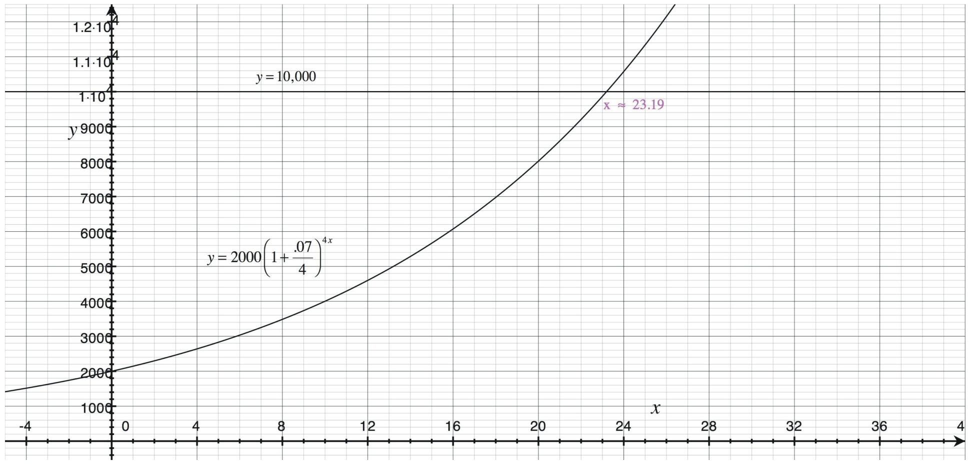
r	$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$	e^r
0		
1		
2		
3		
4		
5		
10		

12.

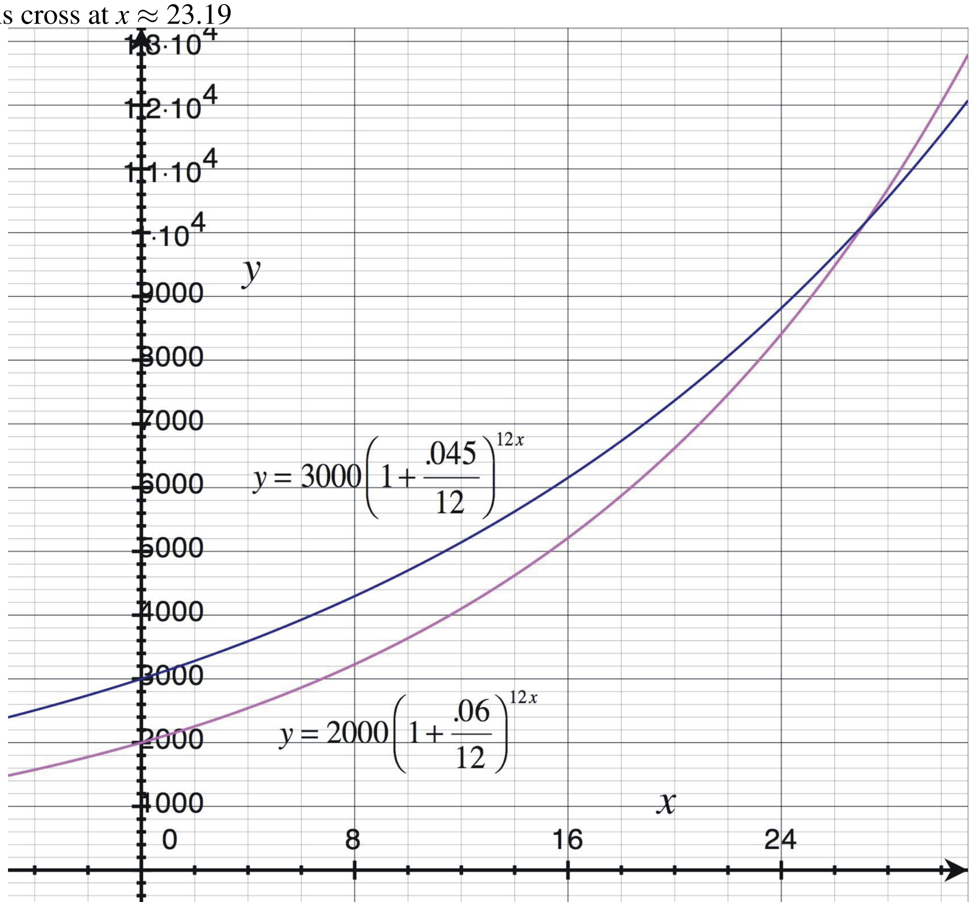
Review Answers

1. After 2 years: \$3905.99. After 5 years: \$4604.96.
2. \$4003.20

3. $t = \frac{ln5}{4ln1.0175} \approx 23.19$ years.



4. The functions cross at $x \approx 23.19$



5. It takes about 27 years for the two investments to have the same value.
6. $t = \frac{ln2}{12ln1.005} \approx 11.58$ years.
7. When solving for t, the 6000 is divided by 3000, resulting in a 2 on the left side of the equation. (Hence the ln 2.) This would be the same, no matter what the initial investment was.
8. \$4694.03
9. \$9892.36
10. It will take about 50 years.

TABLE 5.55:

r	$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$	e^r
0	1	1

TABLE 5.55: (continued)

r	$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r$	e^r
1	e	e
2	≈ 7.389	≈ 7.389
3	≈ 20.086	≈ 20.086
4	≈ 54.598	≈ 54.598
5	≈ 148.413	≈ 148.413
10	≈ 22026.466	≈ 22026.466

11.
12. The values in the table match, but this does not count as a proof. A proof needs to show that the values match for ALL values of
13. r
14. .

Vocabulary

Accrue To accrue is to increase in amount or value over time. If interest accrues on a bank account, you will have more money in your account. If interest accrues on a loan, you will owe more money to your lender.

Compound interest Compound interest is interest based on a principal and on previous interest earned.

Continuous compounding Interest that is based on continuous compounding is calculated according to the equation $A(t)=Pe^{rt}$, where P is the principal, r is the interest rate, t is the length of the investment, and A is the value of the account or investment after t years.

Principal The principal is the initial amount of an investment or a loan.

Simple interest Simple interest is interest that is calculated as a percent of the principal, as a function of time.

5.7 Growth and Decay

Learning objectives

- Model situations using exponential and logistic functions.
- Solve problems involving these models, using your knowledge of properties of logarithms, and using a graphing calculator.

Introduction

In lesson 5 you learned about modeling phenomena with exponential and logarithmic functions. In the examples in lesson 5, you used a graphing calculator to find a line that fits a given set of data. Here we will use algebraic techniques to develop models, and you will learn about another kind of function, the logistic function, that can be used to model growth.

Exponential growth

In general, if you have enough information about a situation, you can write an exponential function to model growth in the situation. Lets start with a straightforward example:

Example 1: A social networking website is started by a group of 10 friends. They advertise their site before they launch, and membership grows fast: the membership doubles every day. At this rate, what will the membership be in a week? When will the membership reach 100,000?

Solution: To model this situation, lets look at how the membership changes each day:

TABLE 5.56:

Time (in days)	Membership
0	10
1	2 10 = 20
2	2 2 10 = 40
3	2 2 2 10 = 80
4	2 2 2 2 10 = 160

Notice that the membership on day x is $10(2^x)$. Therefore we can model membership with the function $M(x) = 10(2^x)$. In seven days, the membership will be $M(7) = 10(2^7) = 1280$.

We can solve an exponential equation to find out when the membership will reach 100,000:

$$10(2^x) = 100,000\ 2^x = 10,000\ log2^x = log10,000\ xlog2 = 4\ x = \frac{4}{log2} \approx 13.3$$

At this rate, the membership will reach 100,000 in less than two weeks. This result may seem unreasonable. Thats very fast growth!

So lets consider a slower rate of doubling. Lets say that the membership doubles every 7 days.

TABLE 5.57:

Time (in days)	Membership
0	10
7	2 10 = 20
14	2 2 10 = 40
21	2 2 2 10 = 80
28	2 2 2 2 10 = 160

We can no longer use the function $M(x) = 10(2^x)$. However, we *can* use this function to find another function to model this new situation. Looking at one data point will help. Consider for example the fact that $M(21) = 10(2^3)$. This is the case because 21 days results in 3 periods of doubling. In order for $x = 21$ to produce 2^3 in the equation, the exponent in the function must be $x/7$. So we have $M(x) = 10\left(2^{\frac{x}{7}}\right)$. Lets verify that this equation makes sense for the data in the table:

$$M(0) = 10\left(2^{\frac{0}{7}}\right) = 10(1) = 10\ M(7) = 10\left(2^{\frac{7}{7}}\right) = 10(2) = 20\ M(14) = 10\left(2^{\frac{14}{7}}\right) = 10(2^2) = 10(4) = 40\ M(21) = 10\left(2^{\frac{21}{7}}\right) = 10(2^3) = 10(8) = 80\ M(28) = 10\left(2^{\frac{28}{7}}\right) = 10(2^4) = 10(16) = 160$$

Notice that each x value represents one more event of doubling, and in order for the function to have the correct power of 2, the exponent must be $(x/7)$.

With the new function $M(x) = 10\left(2^{\frac{x}{7}}\right)$, the membership doubles to 20 in one week, and reaches 100,000 in about 3 months:

$$10\left(2^{\frac{x}{7}}\right) = 100,000\ 2^{\frac{x}{7}} = 100,000\ log2^{\frac{x}{7}} = log100,000\ \tfrac{x}{7}log2 = 4\ xlog2 = 28\ x = \tfrac{28}{log2} \approx 93$$

The previous two examples of exponential growth have specifically been about doubling. We can also model a more general growth pattern with a more general growth model. While the graphing calculator produces a function of the form $y = a(b^x)$, population growth is often modeled with a function in which e is the base. Lets look at this kind of example:

The population of a town was 20,000 in 1990. Because of its proximity to technology companies, the population grew to 35,000 by the year 2000. If the growth continues at this rate, how long will it take for the population to reach 1 million?

The general form of the exponential growth model is much like the continuous compounding function you learned in the previous lesson. We can model exponential growth with a function of the form $P(\mathrm{t}) = P_0e^{kt}$. The expression $P(t)$ represents the population after t years, the coefficient P_0 represents the initial population, and k is a growth constant that depends on the particular situation.

In the situation above, we know that P_0 = 20,000 and that $P(10)$ = 35, 000. We can use this information to find the value of k:

$$P(t) = P_0e^{kt}\ P(10) = 35000 = 20000e^{k\cdot 10}\ \tfrac{35,000}{20000} = e^{10k}\ 1.75 = e^{10k}\ ln1.75 = lne^{10k}\ ln1.75 = 10klne\ ln1.75 = 10k(1)\ ln1.75 = 10k\ k = \tfrac{ln1.75}{10} \approx 0.056$$

Therefore we can model the population growth with the function $P(t) = 20000e^{\frac{ln1.75}{10}t}$. We can determine when the population will reach 1,000,000 by solving an equation, or using a graph.

Here is a solution using an equation: $1000000 = 20000e^{\frac{ln1.75}{10}t}\ 50 = e^{\frac{ln1.75}{10}t}\ ln50 = ln\left(e^{\frac{ln1.75}{10}t}\right)\ ln50 = \frac{ln1.75}{10}t(lne)\ ln50 = \frac{ln1.75}{10}t(1)\ 10ln50 = ln1.75t\ t = \frac{10ln50}{ln1.75} \approx 70$

At this rate, it would take about 70 years for the population to reach 1 million. Like the initial doubling example, the growth rate may seem very fast. In reality, a population that grows exponentially may not sustain its growth rate over time. Next we will look at a different kind of function that can be used to model growth of this kind.

Logistic models

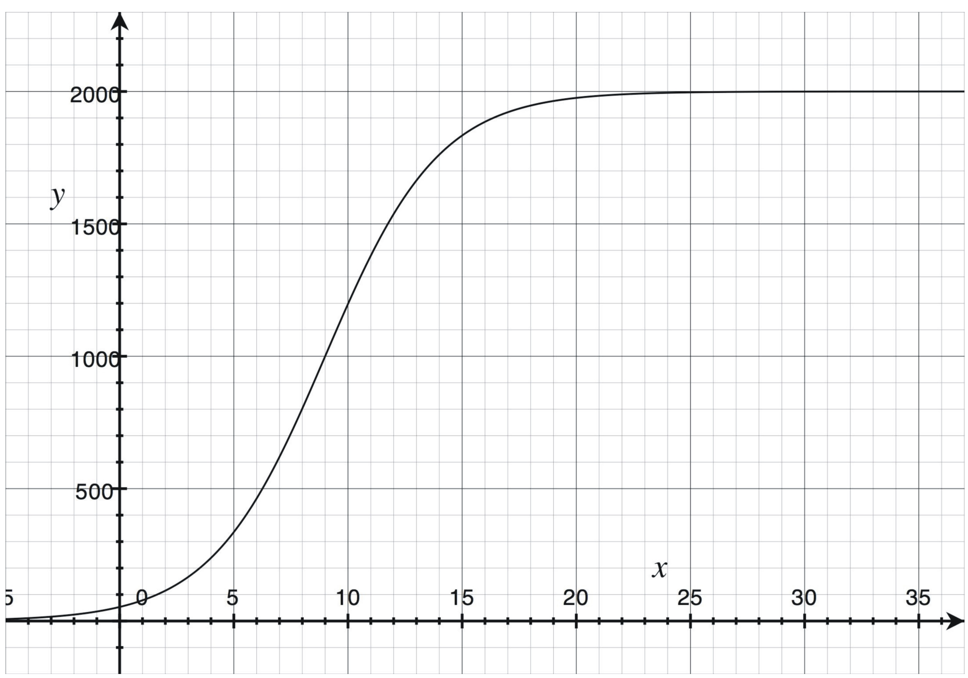
Given that resources are limited, a population may slow down in its growth over time. Consider the last example, the town whose population exploded in the 1990s. If there are no more houses to be bought, or tracts of land to be developed, the population will not continue to grow exponentially. The table below shows the population of this town slowing down, though still growing:

TABLE 5.58:

t (1990=0)	Population
0	20,000
10	35,000
15	38,000
20	40,000

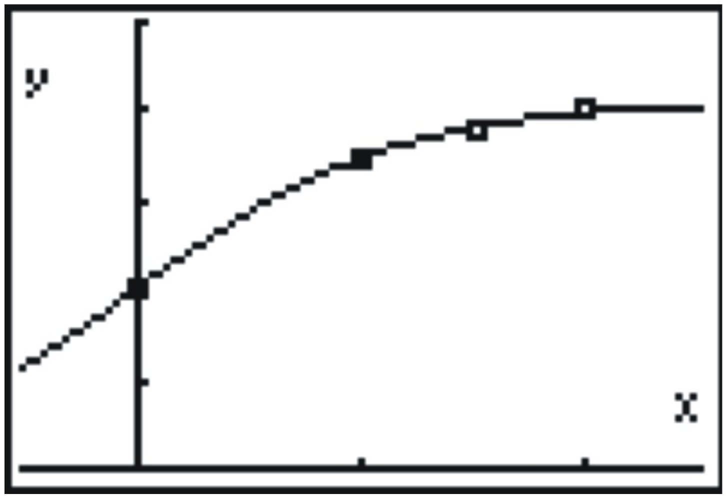
As the population growth slows down, the population may approach what is called a **carrying capacity,** or an upper bound of the population. We can model this kind of growth using a logistic function, which is a function of the form $f(x) = \frac{c}{1+a(e^{-bx})}$.

The graph below shows an example of a logistic function. This kind of graph is often called an s curve because of its shape.



Notice that the graph shows slow growth, then fast growth, and then slow growth again, as the population or quantity in question approaches the carrying capacity. Logistics functions are used to model population growth, as well as other situations, such as the amount of medicine in a persons system

Given the population data above, we can use a graphing calculator to find a logistic function to model this situation. The details of this process are explained in the Technology Note in Lesson 3.5. As shown there, enter the data into L_1 and L_2. Then run a logistic regression. (Press <TI font_STAT>, scroll right to CALC, and scroll down to B. Logistic.) An approximation of the logistic model for this data is: $f(x) = \frac{41042.38}{1+1.050e^{-.178x}}$. A graph of this function and the data is shown here.



Notice that the graph has a horizontal asymptote around 40,000. Looking at the equation, you should notice that the numerator is about 41,042. This value is in fact the horizontal asymptote, which represents the carrying capacity. We can understand why this is the carrying capacity if we consider the limit of the function as x approaches infinity. As x gets larger and larger, $e^{-.178x}$ will get smaller and smaller. So 1.05 $e^{-.178x}$ will get smaller. This means that the denominator of the function will get closer and closer to 1:

$\lim_{x \to \infty}(1+1.050e^{-.178x}) = 1.$

Therefore the limit of the function is (approximately) (41042/1) = 41042. This means that given the current growth, the model predicts that the population will not go beyond 41,042. This kind of growth is seen in population, as well as other situations in which some quantity grows very fast and then slows down, or when a quantity steeply decreases, and then levels off. You will see work with more examples of logistic functions in the review questions.

Exponential decay

Just as a quantity can grow, or increase exponentially, we can model a decreasing quantity with an exponential function. This kind of situation is referred to as exponential decay. Perhaps the most common example of exponential decay is that of radioactive decay, which refers to the transformation of an atom of one type into an atom of a different type, when the nucleus of the atom loses energy. The rate of radioactive decay is usually measured in terms of half-life, or the time it takes for half of the atoms in a sample to decay. For example Carbon-14 is a radioactive isotope that is used in carbon dating, a method of determining the age of organic materials. The half-life of Carbon-14 is 5730 years. This means that if we have a sample of Carbon-14, it will take 5730 years for half of the sample to decay. Then it will take another 5730 years for half of the remaining sample to decay, and so on.

We can model decay using the same form of equation we use to model growth, except that the exponent in the equation is negative: $A(t) = A_0e^{-kt}$. For example, say we have a sample of Carbon-14. How much time will pass before 75% of the original sample remains?

We can use the half-life of 5730 years to determine the value of k:

TABLE 5.59:

$A(t) = A_0e^{-kt}$ $\frac{1}{2} = 1e^{-k \cdot 5730}$	We do not know the value of A_0, *so we use 1 as 100%.* *(1/2) of the sample remains when* t = *5730 years*
$ln\frac{1}{2} = lne^{-k \cdot 5730}$	Take the ln of both sides
$ln\frac{1}{2} = -5730klne$	Use the power property of logs
$ln\frac{1}{2} = -5730k$	$ln(e) = 1$

TABLE 5.59: (continued)

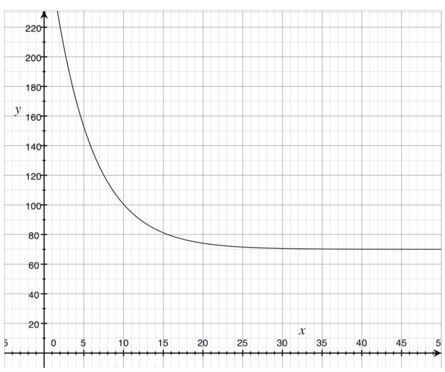
$A(t) = A_0 e^{-kt}$	
$-ln2 = -5730k$	$ln(1/2) = ln(2^{-1}) = -ln2$
$ln2 = 5730k$	
$k = \frac{ln2}{5730}$	Isolate k

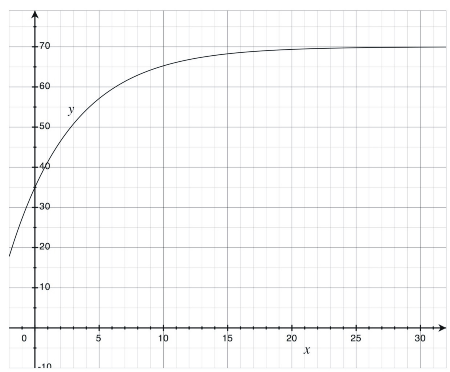
Now we can determine when the amount of Carbon-14 remaining is 75% of the original:

$$0.75 = 1e^{\frac{-ln2}{5730}t}\ 0.75 = 1e^{\frac{-ln2}{5730}t}\ ln(0.75) = lne^{\frac{-ln2}{5730}t}\ ln(0.75) = \frac{-ln2}{5730}t\ t = \frac{5730 ln(0.75)}{-ln2} \approx 2378$$

Therefore it would take about 2,378 years for 75% of the original sample to be remaining. In practice, scientists can approximate the age of an artifact using a process that relies on their knowledge of the half-life of Carbon-14, as well as the ratio of Carbon-14 to Carbon-12 (the most abundant, stable form of carbon) in an object. While the concept of half-life often is used in the context of radioactive decay, it is also used in other situations. In the review questions, you will see another common example, that of medicine in a persons system.

Related to exponential decay is Newtons Law of Cooling. The Law of Cooling allows us to determine the temperature of a cooling (or warming) object, based on the temperature of the surroundings and the time since the object entered the surroundings. The general form of the cooling function is $T(x) = T_5 + (T_0 - T_5)\ e^{-kx}$, where T_5, is the surrounding temperature, T_0 is the initial temperature, and x represents the time since the object began cooling or warming.





The first graph shows a situation in which an object is cooling. The graph has a horizontal asymptote at $y = 70$. This tells us that the object is cooling to 70°F. The second graph has a horizontal asymptote at $y = 70$ as well, but in this situation, the object is warming up to 70°F.

We can use the general form of the function to answer questions about cooling (or warming) situations. Consider the following example: you are baking a casserole in a dish, and the oven is set to 325°F. You take the pan out of the oven and put it on a cooling rack in your kitchen which is 70°F, and after 10 minutes the pan has cooled to 300°F. How long will it take for the pan to cool to 200^0F?

We can use the general form of the equation and the information given in the problem to find the value of k:

$$T(x) = T_s + (T_0 - T_s)e^{-kx}\ T(x) = 70 + (325 - 70)e^{-kx}\ T(x) = 70 + (255)e^{-kx}\ T(10) = 70 + 255e^{-10k} = 300\ 255e^{-10k} = 230\ e^{-10k} = \frac{230}{255}\ lne^{-10k} = ln\left(\frac{230}{255}\right)\ -10k = ln\left(\frac{230}{255}\right)\ k = \frac{ln\left(\frac{230}{255}\right)}{-10} \approx 0.0103$$

Now we can determine the amount of time it takes for the pan to cool to 200 degrees:

$$T(x) = 70 + (255)e^{-.0103}x\ T(x) = 70 + (255)e^{-.0103}x\ 200 = 70 + (255)e^{-.0103}x\ 130 = (255)e^{-.0103}x\ \frac{130}{255} = e^{-.0103}x\ ln\left(\frac{130}{255}\right) = lne^{-.0103}x\ ln\left(\frac{130}{255}\right) = -.0103x\ x = \frac{ln\left(\frac{130}{255}\right)}{-.0103} \approx 65$$

Therefore, in the given surroundings, it would take about an hour for the pan to cool to 200 degrees.

Lesson Summary

In this lesson we have developed exponential and logistic models to represent different phenomena. We have considered exponential growth, logistic growth, and exponential decay. After reading the examples in this lesson, you should be able to write a function to represent a given situation, to evaluate the function for a given value of x, and to solve exponential equations in order to find values of x, given values of the function. For example, in a situation of exponential population growth as a function of time, you should be able to determine the population at a particular time, and to determine the time it takes for the population to reach a given amount. You should be able to solve these kinds of problems by solving exponential equations, and by using graphing utilities, as we have done throughout the chapter.

Points to Consider

1. How can we use the same equation for exponential growth and decay?
2. What are the restrictions on domain and range for the examples in this lesson?
3. How can we use different equations to model the same situations?

Review Questions

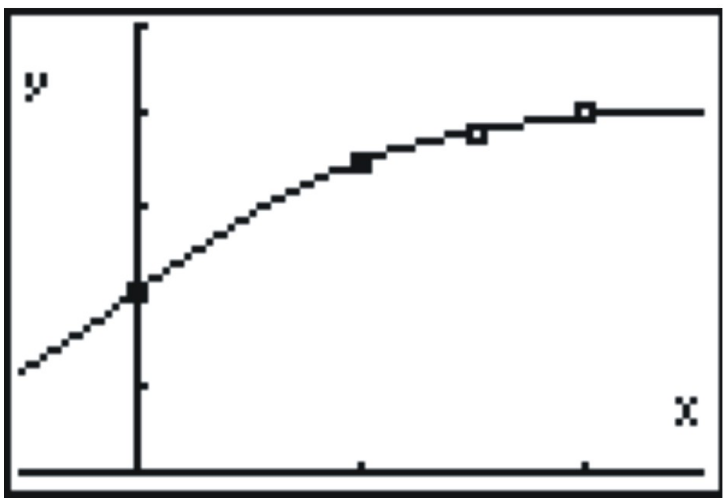
1. The population of a town was 50,000 in 1980, and it grew to 70,000 by 1995.
   a. Write an exponential function to model the growth of the population.
   b. Use the function to estimate the population in 2010.
   c. What if the population growth was linear? Write a linear equation to model the population growth, and use it to estimate the population in 2010.
2. A telecommunications company began providing wireless service in 1994, and during that year the company had 1000 subscribers. By 2004, the company had 12,000 subscribers.
   a. Write an exponential function to model the situation
   b. Use the model to determine how long it will take for the company to reach 50,000 subscribers.
3. The population of a particular strain of bacteria triples every 8 hours.
   a. Write a general exponential function to model the bacteria growth.
   b. Use the model to determine how long it will take for a sample of bacteria to be 100 times its original population.
   c. Use a graph to verify your solution to part b.
4. The half-life of acetaminophen is about 2 hours.
   a. If you take 650 mg of acetaminophen, how much will be left in your system after 7 hours?
   b. How long before there is less than 25 mg in your system?
5. The population of a city was 200,000 in 1991, and it decreased to 170,000 by 2001.
   a. Write an exponential function to model the decreasing population, and use the model to predict the population in 2008.
   b. Under what circumstances might the function cease to model the situation after a certain point in time?
6. Consider the following situation: you buy a large box of pens for the start of the school year, and after six weeks, (1/3) of the pens remain. After another six weeks, (1/3) of the remaining pens were remaining. If you continue this pattern, when will you only have 5% of the pens left?
7. Use Newtons law of cooling to answer the question: you pour hot water into a mug to make tea. The temperature of the water is about 200 degrees. The surrounding temperature is about 75 degrees. You let the water cool for 5 minutes, and the temperature decreases to 160 degrees. What will the temperature be after 15 minutes?
8. The spread of a particular virus can be modeled with the logistic function $f(x) = \frac{2000}{1+600e^{-.75x}}$, where x is the number of days the virus has been spreading, and f(x) represents the number of people who have the virus.
   a. How many people will be affected after 7 days?

b. How many days will it take for the spread to be within one person of carrying capacity?

9. Consider again the situation in problem #2: A telecommunications company began providing wireless service in 1994, and during that year the company had 1000 subscribers. By 2004, the company had 12,000 subscribers. If the company has 15,000 subscribers in 2005, and 16,000 in 2007, what type of model do you think should be used to model the situation? Use a graphing calculator to find a regression equation, and use the equation to predict the number of subscribers in 2010.
10. Compare exponential and logistic functions as tools for modeling growth. What do they have in common, and how do they differ?

Review Answers

1. a. $A(T) = 50,000e^{\frac{ln(\frac{7}{5})}{15}t}$
   b. $98,000$
   c. $f(t) = \frac{4000}{3}t + 50000$. The population would be 90,000, which is different by about 9%.
2. a. $S(t) = 1000e^{\frac{ln(12)}{10}t}$
   b. $t = \frac{10ln50}{ln12} \approx 15.74$
3. a. $A(t) = A_0\ (3^{t/8})$
   b. $t = \frac{16}{log3} \approx 33.53$
   c. The graph below shows $y = 100$ and $y = 3^{x/8}$, which intersect at approximately $x = 33.53$



4. a. About 57.45 mg
   b. About 9.4 hours
5. a. $P(t) = 200000e^{\frac{-ln.85}{-10}t}, P(17) \approx 151720$
   b. If the economy or other factors change, the population might begin to increase, or the rate of decrease could change as well.
6. $t = \frac{6log(0.05)}{log(\frac{1}{3})} \approx 16$ weeks
7. About 114 degrees.
8. a. About 482 people.
   b. After 19 days, over 1999 people have the virus.
9. The graph indicates a logistic model. $f(x) \approx \frac{18872}{1+21.45e^{-377x}}$ gives 17952 subscribers in 2010.
10. Both types of functions model fast increase in growth, but the logistic model shows the growth slowing down after some point, with some upper bound on the quantity in question. (Many people argue that logistic growth is more realistic.)

Vocabulary

Carrying capacity The supportable population of an organism, given the food, habitat, water and other necessities available within an ecosystem is known as the ecosystem's carrying capacity for that organism.

Radioactive decay Radioactive decay is the process in which an unstable atomic nucleus loses energy by emitting radiation in the form of particles or electromagnetic waves. This decay, or loss of energy, results in an atom of one type transforming to an atom of a different type. For example, Carbon-14 transforms into Nitrogen-14

Half-life The amount of time it takes for half of a given amount of a substance to decay. The half-life remains the same, no matter how much of the substance there is.

Isotope Isotopes are any of two or more forms of a chemical element, having the same number of protons in the nucleus, or the same atomic number, but having different numbers of neutrons in the nucleus, or different atomic weights.

5.8 Applications

Learning objectives

- Work with the decibel system for measuring loudness of sound.
- Work with the Richter scale, which measures the magnitude of earthquakes.
- Work with pH values and concentrations of hydrogen ions.

Introduction

Because logarithms are related to exponential relationships, logarithms are useful for measuring phenomena that involve very large numbers or very small numbers. In this lesson you will learn about three situations in which a quantity is measured using logarithms. In each situation, a logarithm is used to simplify measurements of either very small numbers or very large numbers. We begin with measuring the intensity of sound.

Intensity of sound

Sound intensity is measured using a logarithmic scale. The intensity of a sound wave is measured in Watts per square meter, or W/m^2. Our hearing threshold (or the minimum intensity we can hear at a frequency of 1000 Hz), is 2.5 $10^{12} W/m^2$. The intensity of sound is often measured using the decibel (dB) system. We can think of this system as a function. The input of the function is the intensity of the sound, and the output is some number of decibels. The decibel is a dimensionless unit; however, because decibels are used in common and scientific discussions of sound, the values of the scale have become familiar to people.

We can calculate the decibel measure as follows:

$$\text{Intensity level } (dB) = 10log\left[\frac{\text{intensity of sound in } W/m^2}{.937\times10^{-12}W/m^2}\right]$$

An intensity of .937 $10^{-12}W/m^2$ corresponds to 0 decibels:

$$10log\left[\frac{.937\times10^{-12}W/m^2}{.937\times10^{-12}W/m^2}\right] = 10log1 = 10(0) = 0.$$

Note: The sound equivalent to 0 decibels is approximately the lowest sound that humans can hear. If the intensity is ten times as large, the decibel level is 10:

$$10log\left[\frac{.937\times10^{-11}W/m^2}{.937\times10^{-12}W/m^2}\right] = 10log10 = 10(1) = 10$$

If the intensity is 100 times as large, the decibel level is 20, and if the intensity is 1000 times as large, the decibel level is 30. (The scale is created this way in order to correspond to human hearing. We tend to underestimate intensity.) The threshold for pain caused by sound is 1 W/m^2. This intensity corresponds to about 120 decibels:

$$10log\left[\frac{1W/m^2}{.937\times10^{-12}}W/m^2\right] \approx 10:12 = 120$$

Many common phenomena are louder than this. For example, a jet can reach about 140 decibels, and concert can reach about 150 decibels.

(Source: Ohanian, H.C. (1989) Physics. New York: W.W. Norton Company.)

For ease of calculation, the equation is often simplified: .937 is rounded to 1:

TABLE 5.60:

Intensity level (dB)	$= 10log\left[\frac{\text{intensity of sound in } W/m^2}{1\times10^{-12}W/m^2}\right]$ $= 10log\left[\frac{\text{intensity of sound in } W/m^2}{10^{-12}W/m^2}\right]$

In the example below we will use this simplified equation to answer a question about decibels. (In the review exercises, you can also use this simplified equation).

Example 1: Verify that a sound of intensity 100 times that of a sound of 0 dB corresponds to 20 dB.

Solution: $dB = 10log\left(\frac{100\times10^{-12}}{10^{-12}}\right) = 10log(100) = 10(2) = 20.$

Intensity and magnitude of earthquakes

An earthquake occurs when energy is released from within the earth, often caused by movement along fault lines. An earthquake can be measured in terms of its intensity, or its magnitude. Intensity refers to the effect of the earthquake, which depends on location with respect to the epicenter of the quake. Intensity and magnitude are not the same thing.

As mentioned in lesson 3, the magnitude of an earthquake is measured using logarithms. In 1935, scientist Charles Richter developed this scale in order to compare the size of earthquakes. You can think of Richter scale as a function in which the input is the amplitude of a seismic wave, as measured by a seismograph, and the output is a magnitude. However, there is more than one way to calculate the magnitude of an earthquake because earthquakes produce two different kinds of waves that can be measured for amplitude. The calculations are further complicated by the need for a correction factor, which is a function of the distance between the epicenter and the location of the seismograph.

Given these complexities, seismologists may use different formulas, depending on the conditions of a specific earthquake. This is done so that the measurement of the magnitude of a specific earthquake is consistent with Richters original definition. (Source: http://earthquake.usgs.gov/learning/topics/richter.php)

Even without a specific formula, we can use the Richter scale to compare the size of earthquakes. For example, the 1906 San Francisco earthquake had a magnitude of about 7.7. The 1989 Loma Prieta earthquake had a magnitude of about 6.9. (The epicenter of the quake was near Loma Prieta peak in the Santa Cruz mountains, south of San Francisco.) Because the Richter scale is logarithmic, this means that the 1906 quake was six times as strong as the 1989 quake:

$$\frac{10^{7.7}}{10^{6.9}} = 10^{7.7-6.9} = 10^{.8} \approx 6.3$$

This kind of calculation explains why magnitudes are reported using a whole number and a decimal. In fact, a decimal difference makes a big difference in the size of the earthquake, as shown below and in the review exercises

Example 2: An earthquake has a magnitude of 3.5. A second earthquake is 100 times as strong. What is the magnitude of the second earthquake?

Solution: The second earthquake is 100 times as strong as the earthquake of magnitude 3.5. This means that if the magnitude of the second earthquake is x, then:

$$\frac{10^x}{10^{3.5}} = 100 \; 10^{x-3.5} = 100 = 10^2 \; x-3.5 = 2 \; x = 5.5$$

So the magnitude of the second earthquake is 5.5.

The pH scale

If you have studied chemistry, you may have learned about acids and bases. An acid is a substance that produces hydrogen ions when added to water. A hydrogen ion is a positively charged atom of hydrogen, written as H^+. A base is a substance that produces hydroxide ions (OH^-) when added to water. Acids and bases play important roles in everyday life, including within the human body. For example, our stomachs produce acids in order to breakdown foods. However, for people who suffer from gastric reflux, acids travel up to and can damage the esophagus. Substances that are bases are often used in cleaners, but a strong base is dangerous: it can burn your skin.

To measure the concentration of an acid or a base in a substance, we use the pH scale, which was invented in the early 1900s by a Danish scientist named Soren Sorenson. The pH of a substance depends on the concentration of H^+, which is written with the symbol $[H^+]$.

pH = - log $[H^+]$

(Note: concentration is usually measured in moles per liter. A mole is 6.02 10^{23} units. Here, it would be 6.02 10^{23} hydrogen ions.)

For example, the concentration of H^+ in stomach acid is about 1 10^{-1}. So the pH of stomach acid is -log (10^{-1}) = -(-1) = 1. The pH scale ranges from 0 to 14. A substance with a low pH is an acid. A substance with a high pH is a base. A substance with a pH in the middle of the scale is considered to be neutral.

Example 3: The pH of ammonia is 11. What is the concentration of H^+?

Solution: pH = - log $[H^+]$. If we substitute 11 for pH we can solve for H^+:

$11 = -log[H^+]$ $-11 = log[H^+]$ $10^{-11} = 10^{log[H^+]}$ $10^{-11} = H^+$

Lesson Summary

In this lesson we have looked at three examples of logarithmic scales. In the case of the decibel system, using a logarithm has produced a simple way of categorizing the intensity of sound. The Richter scale allows us to compare earthquakes. And, the pH scale allows us to categorize acids and bases. In each case, a logarithm helps us work with large or small numbers, in order to more easily understand the quantities involved in certain real world phenomena.

Points to Consider

1. How are the decibel system and the Richter scale the same, and how are they different?
2. What other phenomena might be modeled using a logarithmic scale?

Review Questions

1. Verify that a sound of intensity 1000 times that of a sound of 0 dB corresponds to 30 dB.
2. Calculate the decibel level of a sound with intensity $10^{-8}W/m^2$.
3. Calculate the intensity of a sound if the decibel level is 25.
4. The 2004 Indian Ocean earthquake was recorded to have a magnitude of about 9.5. In 1960, an earthquake in Chile was recorded to have a magnitude of 9.1. How much stronger was the 2004 Indian Ocean quake?
5. Two earthquakes of the same magnitude do not necessarily cause the same amount of destruction. How is that possible?
6. The concentration of H^+ in pure water is 1 10^{-7}. What is the pH?
7. The pH of normal human blood is 7.4. What is the concentration of H^+

Review Answers

1. $dB = 10log\left(\frac{100\times10^{-12}}{10^{-12}}\right) = 10log(100) = 10(3) = 30$
2. $dB = 10log\left(\frac{10^{-8}}{10^{-12}}\right) = 10log(10000) = 10(4) = 40$

3. $10^{-9.5}$ or $3.16\ 10^{-10}$
4. $10^{0.4} \approx 2.5$
5. According to the USGS, the damage depends on the strength of shaking, the length of shaking, the type of soil in the area, and the types of buildings. Many buildings in the San Francisco Bay Area are undergoing seismic retrofitting, in anticipation of the big one.
6. The pH is 7.
7. $10^{-7.4} \approx 3.98\ 10^{-8}$

Vocabulary

Acid An acid is a substance that produces hydrogen ions when added to water.

Amplitude The amplitude of a wave is the distance from its highest (or lowest) point to its center.

Base A base is a substance that produces hydroxide ions (OH^-) when added to water

Decibel A decibel is a unitless measure of the intensity of sound.

Mole $6.02\ 10^{23}$ units of a substance.

Seismograph A seismograph is a device used to measure the amplitude of earthquakes.