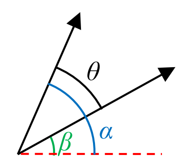
Section 8.5 Dot Product

Now that we can add, subtract, and scale vectors, you might be wondering whether we can multiply vectors. It turns out there are two different ways to multiply vectors, one which results in a number, and one which results in a vector. In this section, we'll focus on the first, called the **dot product** or **scalar product**, since it produces a single numeric value (a scalar). We'll begin with some motivation.

In physics, we often want to know how much of a force is acting in the direction of motion. To determine this, we need to know the angle between direction of force and the direction of motion. Likewise, in computer graphics, the lighting system determines how bright a triangle on the object should be based on the angle between object and the direction of the light. In both applications, we're interested in the angle between the vectors, so let's start there.

Suppose we have two vectors, $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$. Using our polar coordinate conversions, we could write $\vec{a} = \langle |\vec{a}|\cos(\alpha), |\vec{a}|\sin(\alpha) \rangle$ and $\vec{b} = \left\langle |\vec{b}|\cos(\beta), |\vec{b}|\sin(\beta) \right\rangle$.



Now, if we knew the angles α and β, we wouldn't have much work to do - the angle between the vectors would be $\theta = \alpha - \beta$. While we certainly could use some inverse tangents to find the two angles, it would be great if we could find a way to determine the angle between the vector just from the vector components.

To help us manipulate $\theta = \alpha - \beta$, we might try introducing a trigonometric function:

$$\cos(\theta) = \cos(\alpha - \beta)$$

Now we can apply the difference of angles identity

$$\cos(\theta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

Now $a_1 = |\vec{a}|\cos(\alpha)$, so $\cos(\alpha) = \dfrac{a_1}{|\vec{a}|}$, and likewise for the other three components.

Making those substitutions,

$$\cos(\theta) = \frac{a_1}{|\vec{a}|}\frac{b_1}{|\vec{b}|} + \frac{a_2}{|\vec{a}|}\frac{b_2}{|\vec{b}|} = \frac{a_1 b_1 + a_2 b_2}{|\vec{a}||\vec{b}|}$$

$$|\vec{a}||\vec{b}|\cos(\theta) = a_1 b_1 + a_2 b_2$$

Notice the expression on the right is a very simple calculation based on the components of the vectors. It comes up so frequently we define it to be the **dot product** of the two vectors, notated by a dot. This gives us two definitions of the dot product.

Definitions of the Dot Product	
$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$	Component definition
$\vec{a} \cdot \vec{b} = \lvert\vec{a}\rvert \lvert\vec{b}\rvert \cos(\theta)$	Geometric definition

The first definition, $\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$, gives us a simple way to calculate the dot product from components. The second definition, $\vec{a} \cdot \vec{b} = \lvert\vec{a}\rvert \lvert\vec{b}\rvert \cos(\theta)$, gives us a geometric interpretation of the dot product, and gives us a way to find the angle between two vectors, as we desired.

Example 1

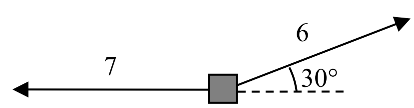
Find the dot product $\langle 3,-2\rangle \cdot \langle 5,1\rangle$.

Using the first definition, we can calculate the dot product by multiplying the x components and adding that to the product of the y components.

$$\langle 3,-2\rangle \cdot \langle 5,1\rangle = (3)(5) + (-2)(1) = 15 - 2 = 13$$

Example 2

Find the dot product of the two vectors shown.



We can immediately see that the magnitudes of the two vectors are 7 and 6. We can quickly calculate that the angle between the vectors is 150°. Using the geometric definition of the dot product,

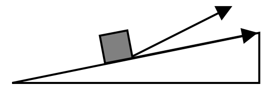
$$\vec{a} \cdot \vec{b} = \lvert\vec{a}\rvert \lvert\vec{b}\rvert \cos(\theta) = (6)(7)\cos(150°) = 42 \cdot \frac{-\sqrt{3}}{2} = -21\sqrt{3}.$$

Try it Now

1. Calculate the dot product $\langle -7,3\rangle \cdot \langle -2,-6\rangle$

Now we can return to our goal of finding the angle between vectors.

Example 3



An object is being pulled up a ramp in the direction $\langle 5,1\rangle$ by a rope pulling in the direction $\langle 4,2\rangle$. What is the angle between the rope and the ramp?

Using the component form, we can easily calculate the dot product.

$$\vec{a}\cdot\vec{b} = \langle 5,1\rangle\cdot\langle 4,2\rangle = (5)(4)+(1)(2) = 20+2 = 22$$

We can also calculate the magnitude of each vector.

$$|\vec{a}| = \sqrt{5^2+1^2} = \sqrt{26}\,,\quad |\vec{b}| = \sqrt{4^2+2^2} = \sqrt{20}$$

Substituting these values into the geometric definition, we can solve for the angle between the vectors.

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$$

$$22 = \sqrt{26}\sqrt{20}\cos(\theta)$$

$$\theta = \cos^{-1}\left(\frac{22}{\sqrt{26}\sqrt{20}}\right) \approx 15.255°.$$

Example 4

Calculate the angle between the vectors $\langle 6,4\rangle$ and $\langle -2,3\rangle$.

Calculating the dot product, $\langle 6,4\rangle\cdot\langle -2,3\rangle = (6)(-2)+(4)(3) = -12+12 = 0$

We don't even need to calculate the magnitudes in this case since the dot product is 0.

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos(\theta)$$

$$0 = |\vec{a}||\vec{b}|\cos(\theta)$$

$$\theta = \cos^{-1}\left(\frac{0}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}(0) = 90°$$

With the dot product equaling zero, as in the last example, the angle between the vectors will always be 90°, indicating that the vectors are **orthogonal**, a more general way of saying perpendicular. This gives us a quick way to check if vectors are orthogonal. Also, if the dot product is positive, then the inside of the inverse cosine will be positive, giving an angle less than 90°. A negative dot product will then lead to an angle larger than 90°

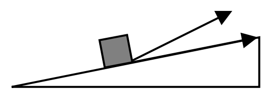
Sign of the Dot Product	
If the dot product is:	
Zero	The vectors are **orthogonal** (perpendicular).
Positive	The angle between the vectors is less than 90°
Negative	The angle between the vectors is greater than 90°

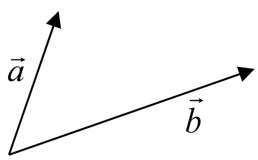
Try it Now

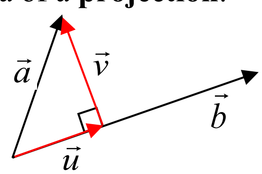
2. Are the vectors $\langle -7,3\rangle$ and $\langle -2,-6\rangle$ orthogonal? If not, find the angle between them.

Projections

In addition to finding the angle between vectors, sometimes we want to know how much one vector points in the direction of another. For example, when pulling an object up a ramp, we might want to know how much of the force is exerted in the direction of motion. To determine this we can use the idea of a **projection**.







In the picture above, $\vec{u}$ is a projection of $\vec{a}$ onto $\vec{b}$. In other words, it is the portion of $\vec{a}$ that points in the same direction as $\vec{b}$.

To find the length of $\vec{u}$, we could notice that it is one side of a right triangle. If we define θ to be the angle between $\vec{a}$ and $\vec{u}$, then $\cos(\theta)=\dfrac{|\vec{u}|}{|\vec{a}|}$, so $|\vec{a}|\cos(\theta)=|\vec{u}|$.

While we could find the angle between the vectors to determine this magnitude, we could skip some steps by using the dot product directly. Since $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos(\theta)$,

$|\vec{a}|\cos(\theta)=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$. Using this, we can rewrite $|\vec{u}|=|\vec{a}|\cos(\theta)$ as $|\vec{u}|=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$. This gives us

the length of the projection, sometimes denoted as $comp_{\vec{b}}\vec{a}=|\vec{u}|=\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$.

To find the vector $\vec{u}$ itself, we could first scale $\vec{b}$ to a unit vector with length 1: $\dfrac{\vec{b}}{|\vec{b}|}$.

Multiplying this by the length of the projection will give a vector in the direction of $\vec{b}$ but with the correct length.

$$proj_{\vec{b}}\vec{a} = |\vec{u}|\frac{\vec{b}}{|\vec{b}|} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\frac{\vec{b}}{|\vec{b}|} = \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

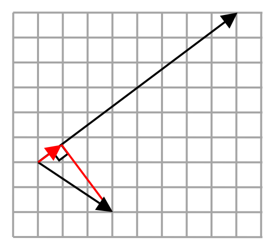
Projection Vector

The projection of vector $\vec{a}$ onto $\vec{b}$ is $proj_{\vec{b}}\vec{a} = \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$

The magnitude of the projection is $comp_{\vec{b}}\vec{a} = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

Example 5

Find the projection of the vector $\langle 3,-2\rangle$ onto the vector $\langle 8,6\rangle$.



We will need to know the dot product of the vectors and the magnitude of the vector we are projecting onto.

$\langle 3,-2\rangle\cdot\langle 8,6\rangle = (3)(8)+(-2)(6) = 24-12 = 12$

$|\langle 8,6\rangle| = \sqrt{8^2+6^2} = \sqrt{64+36} = \sqrt{100} = 10$

The magnitude of the projection will be $\dfrac{\langle 3,-2\rangle\cdot\langle 8,6\rangle}{|\langle 8,6\rangle|} = \dfrac{12}{10} = \dfrac{6}{5}$.

To find the projection vector itself, we would multiply that magnitude by $\langle 8,6\rangle$ scaled to a unit vector.

$$\frac{6}{5}\frac{\langle 8,6\rangle}{|\langle 8,6\rangle|} = \frac{6}{5}\frac{\langle 8,6\rangle}{10} = \frac{6}{50}\langle 8,6\rangle = \left\langle \frac{48}{50},\frac{36}{50}\right\rangle = \left\langle \frac{24}{25},\frac{18}{25}\right\rangle.$$

Based on the sketch above, this answer seems reasonable.

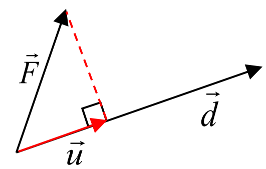
Try it Now

3. Find the component of the vector $\langle -3,4\rangle$ that is *orthogonal* to the vector $\langle -8,4\rangle$

Work

In physics, when a constant force causes an object to move, the mechanical **work** done by that force is the product of the force and the distance the object is moved. However, we only consider the portion of force that is acting in the direction of motion.

This is simply the magnitude of the projection of the force vector onto the distance vector, $\dfrac{\vec{F}\cdot\vec{d}}{\left|\vec{d}\right|}$. The work done is the product of that component of force times the distance moved, the magnitude of the distance vector.



$$Work = \left(\frac{\vec{F}\cdot\vec{d}}{\left|\vec{d}\right|}\right)\left|\vec{d}\right| = \vec{F}\cdot\vec{d}$$

It turns out that work is simply the dot product of the force vector and the distance vector.

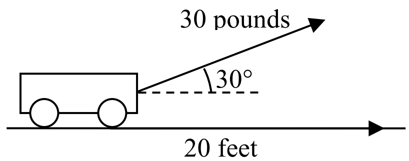
Work

When a force $\vec{F}$ causes an object to move some distance $\vec{d}$, the work done is

$Work = \vec{F}\cdot\vec{d}$

Example 6

A cart is pulled 20 feet by applying a force of 30 pounds on a rope held at a 30 degree angle. How much work is done?



Since work is simply the dot product, we can take advantage of the geometric definition of the dot product in this case.

$Work = \vec{F}\cdot\vec{d} = \left|\vec{F}\right|\cdot\left|\vec{d}\right|\cos(\theta) = (30)(20)\cos(30°) \approx 519.615$ ft-lbs.

Try it Now

4. Find the work down moving an object from the point (1, 5) to (9, 14) by the force vector $\vec{F} = \langle 3,2\rangle$

Important Topics of This Section
Calculate Dot Product
Using component definition
Using geometric definition
Find the angle between two vectors
Sign of the dot product
Projections
Work

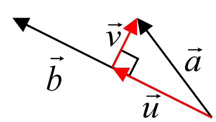
Try it Now Answers

1. $\langle -7,3\rangle \cdot \langle -2,-6\rangle = (-7)(-2)+(3)(-6) = 14-18 = -4$

2. In the previous Try it Now, we found the dot product was -4, so the vectors are not orthogonal. The magnitudes of the vectors are $\sqrt{(-7)^2+3^2} = \sqrt{58}$ and $\sqrt{(-2)^2+6^2} = \sqrt{40}$. The angle between the vectors will be

$$\theta = \cos^{-1}\left(\frac{-4}{\sqrt{58}\sqrt{40}}\right) \approx 94.764°$$

3. We want to find the component of $\langle -3,4\rangle$ that is *orthogonal* to the vector $\langle -8,4\rangle$. In the picture to the right, that component is vector $\vec{v}$. Notice that $\vec{u}+\vec{v}=\vec{a}$, so if we can find the projection vector, we can find $\vec{v}$.



$$\vec{u} = proj_{\vec{b}}\vec{a} = \left(\frac{\vec{a}\cdot\vec{b}}{\left|\vec{b}\right|^2}\right)\vec{b} = \left(\frac{\langle -3,4\rangle\cdot\langle -8,4\rangle}{\left(\sqrt{(-8)^2+4^2}\right)^2}\right)\langle -8,4\rangle = \frac{40}{80}\langle -8,4\rangle = \langle -4,2\rangle.$$

Now we can solve $\vec{u}+\vec{v}=\vec{a}$ for $\vec{v}$.
$\vec{v} = \vec{a}-\vec{u} = \langle -3,4\rangle - \langle -4,2\rangle = \langle 1,2\rangle$

4. The distance vector is $\langle 9-1, 14-5\rangle = \langle 8,9\rangle$.

The work is the dot product: $Work = \vec{F}\cdot\vec{d} = \langle 3,2\rangle\cdot\langle 8,9\rangle = 24+18 = 42$

Section 8.5 Exercises

Two vectors are described by their magnitude and direction in standard position. Find the dot product of the vectors.

1. Magnitude: 6, Direction: 45°; Magnitude: 10, Direction: 120°
2. Magnitude: 8, Direction: 220°; Magnitude: 7, Direction: 305°

Find the dot product of each pair of vectors.

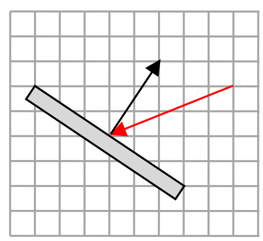
3. $\langle 0,4\rangle$; $\langle -3,0\rangle$
4. $\langle 6,5\rangle$; $\langle 3,7\rangle$
5. $\langle -2,1\rangle$; $\langle -10,13\rangle$
6. $\langle 2,-5\rangle$; $\langle 8,-4\rangle$

Find the angle between the vectors

7. $\langle 0,4\rangle$; $\langle -3,0\rangle$
8. $\langle 6,5\rangle$; $\langle 3,7\rangle$
9. $\langle 2,4\rangle$; $\langle 1,-3\rangle$
10. $\langle -4,1\rangle$; $\langle 8,-2\rangle$
11. $\langle 4,2\rangle$; $\langle 8,4\rangle$
12. $\langle 5,3\rangle$; $\langle -6,10\rangle$

13. Find a value for k so that $\langle 2,7\rangle$ and $\langle k,4\rangle$ will be orthogonal.
14. Find a value for k so that $\langle -3,5\rangle$ and $\langle 2,k\rangle$ will be orthogonal.

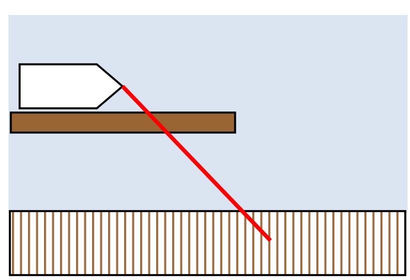
15. Find the magnitude of the projection of $\langle 8,-4\rangle$ onto $\langle 1,-3\rangle$.
16. Find the magnitude of the projection of $\langle 2,7\rangle$ onto $\langle 4,5\rangle$.
17. Find the projection of $\langle -6,10\rangle$ onto $\langle 1,-3\rangle$.
18. Find the projection of $\langle 0,4\rangle$ onto $\langle 3,7\rangle$.



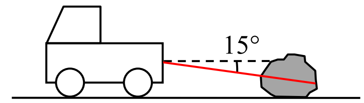
19. A scientist needs to determine the angle of reflection when a laser hits a mirror. The picture shows the vector representing the laser beam, and a vector that is orthogonal to the mirror. Find the acute angle between these, the angle of reflection.

20. A triangle has coordinates at A: (1,4), B: (2,7), and C: (4,2). Find the angle at point B.

21. A boat is trapped behind a log lying parallel to the dock. It only requires 10 pounds of force to pull the boat directly towards you, but because of the log, you'll have to pull at a 45° angle. How much force will you have to pull with? (We're going to assume that the log is very slimy and doesn't contribute any additional resistance.)



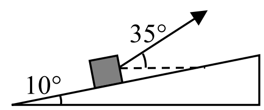
22. A large boulder needs to be dragged to a new position. If pulled directly horizontally, the boulder would require 400 pounds of pulling force to move. We need to pull the boulder using a rope tied to the back of a large truck, forming a 15° angle from the ground. How much force will the truck need to pull with?



23. Find the work done against gravity by pushing a 20 pound cart 10 feet up a ramp that is 10° above horizontal. Assume there is no friction, so the only force is 20 pounds downwards due to gravity.

24. Find the work done against gravity by pushing a 30 pound cart 15 feet up a ramp that is 8° above horizontal. Assume there is no friction, so the only force is 30 pounds downwards due to gravity.

25. An object is pulled to the top of a 40 foot ramp that forms a 10° angle with the ground. It is pulled by rope exerting a force of 120 pounds at a 35° angle relative to the ground. Find the work done.



26. An object is pulled to the top of a 30 foot ramp that forms a 20° angle with the ground. It is pulled by rope exerting a force of 80 pounds at a 30° angle relative to the ground. Find the work done.