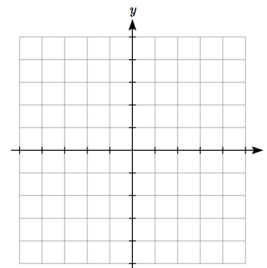


Section 8.2 Polar Coordinates

The coordinate system we are most familiar with is called the Cartesian coordinate system, a rectangular plane divided into four quadrants by horizontal and vertical axes.

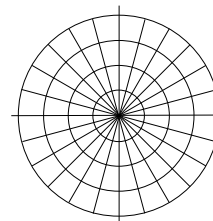
In earlier chapters, we often found the Cartesian coordinates of a point on a circle at a given angle from the positive horizontal axis. Sometimes that angle, along with the point's distance from the origin, provides a more useful way of describing the point's location than conventional Cartesian coordinates.



Polar Coordinates

Polar coordinates of a point consist of an ordered pair, (r, θ) , where r is the distance from the point to the origin, and θ is the angle measured in standard position.

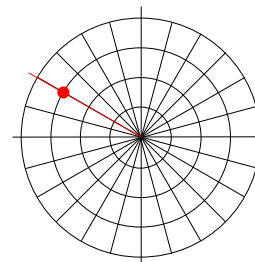
Notice that if we were to “grid” the plane for polar coordinates, it would look like the graph to the right, with circles at incremental radii, and rays drawn at incremental angles.



Example 1

Plot the polar point $\left(3, \frac{5\pi}{6}\right)$.

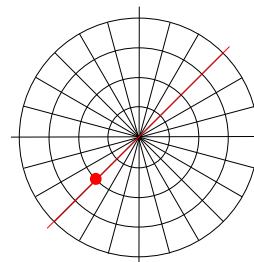
This point will be a distance of 3 from the origin, at an angle of $\frac{5\pi}{6}$. Plotting this,



Example 2

Plot the polar point $\left(-2, \frac{\pi}{4}\right)$.

Typically we use positive r values, but occasionally we run into cases where r is negative. On a regular number line, we measure positive values to the right and negative values to the left. We will plot this point similarly. To start, we rotate to an angle of $\frac{\pi}{4}$. Moving this direction, into the first quadrant, would be positive r values. For negative r values, we move the opposite direction, into the third quadrant. Plotting this:



Note the resulting point is the same as the polar point $\left(2, \frac{5\pi}{4}\right)$. In fact, any Cartesian point can be represented by an infinite number of different polar coordinates by adding or subtracting full rotations to these points. For example, same point could also be represented as $\left(2, \frac{13\pi}{4}\right)$.

Try it Now

1. Plot the following points given in polar coordinates and label them.

a. $A = \left(3, \frac{\pi}{6}\right)$ b. $B = \left(-2, \frac{\pi}{3}\right)$ c. $C = \left(4, \frac{3\pi}{4}\right)$

Converting Points

To convert between polar coordinates and Cartesian coordinates, we recall the relationships we developed back in Chapter 5.

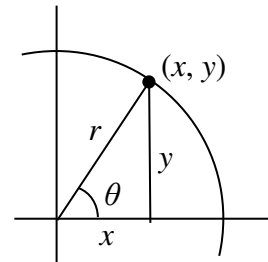
Converting Between Polar and Cartesian Coordinates

To convert between polar (r, θ) and Cartesian (x, y) coordinates, we use the relationships

$$\cos(\theta) = \frac{x}{r} \quad x = r \cos(\theta)$$

$$\sin(\theta) = \frac{y}{r} \quad y = r \sin(\theta)$$

$$\tan(\theta) = \frac{y}{x} \quad x^2 + y^2 = r^2$$



From these relationship and our knowledge of the unit circle, if $r = 1$ and $\theta = \frac{\pi}{3}$, the polar coordinates would be $(r, \theta) = \left(1, \frac{\pi}{3}\right)$, and the corresponding Cartesian coordinates

$$(x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

Remembering your unit circle values will come in very handy as you convert between Cartesian and polar coordinates.

Example 3

Find the Cartesian coordinates of a point with polar coordinates $(r, \theta) = \left(5, \frac{2\pi}{3}\right)$.

To find the x and y coordinates of the point,

$$x = r \cos(\theta) = 5 \cos\left(\frac{2\pi}{3}\right) = 5\left(-\frac{1}{2}\right) = -\frac{5}{2}$$

$$y = r \sin(\theta) = 5 \sin\left(\frac{2\pi}{3}\right) = 5\left(\frac{\sqrt{3}}{2}\right) = \frac{5\sqrt{3}}{2}$$

The Cartesian coordinates are $\left(-\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$.

Example 4

Find the polar coordinates of the point with Cartesian coordinates $(-3, -4)$.

We begin by finding the distance r using the Pythagorean relationship $x^2 + y^2 = r^2$

$$(-3)^2 + (-4)^2 = r^2$$

$$9 + 16 = r^2$$

$$r^2 = 25$$

$$r = 5$$

Now that we know the radius, we can find the angle using any of the three trig relationships. Keep in mind that any of the relationships will produce two solutions on the circle, and we need to consider the quadrant to determine which solution to accept. Using the cosine, for example:

$$\cos(\theta) = \frac{x}{r} = \frac{-3}{5}$$

$$\theta = \cos^{-1}\left(\frac{-3}{5}\right) \approx 2.214$$

By symmetry, there is a second possibility at

$$\theta = 2\pi - 2.214 = 4.069$$

Since the point $(-3, -4)$ is located in the 3rd quadrant, we can determine that the second angle is the one we need. The polar coordinates of this point are $(r, \theta) = (5, 4.069)$.

Try it Now

2. Convert the following.

a. Convert polar coordinates $(r, \theta) = (2, \pi)$ to (x, y) .

b. Convert Cartesian coordinates $(x, y) = (0, -4)$ to (r, θ) .

Polar Equations

Just as a Cartesian equation like $y = x^2$ describes a relationship between x and y values on a Cartesian grid, a polar equation can be written describing a relationship between r and θ values on the polar grid.

Example 5

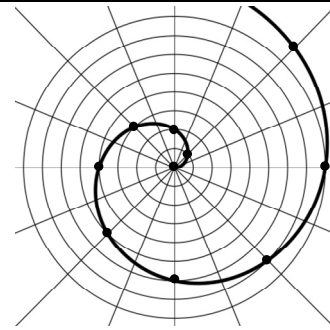
Sketch a graph of the polar equation $r = \theta$.

The equation $r = \theta$ describes all the points for which the radius r is equal to the angle. To visualize this relationship, we can create a table of values.

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
r	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π

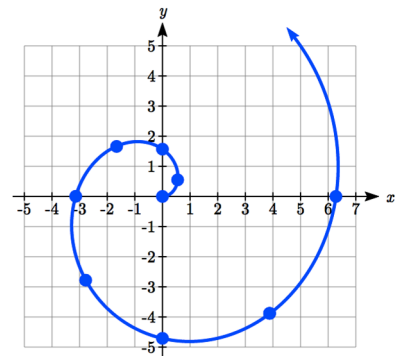
We can plot these points on the plane, and then sketch a curve that fits the points. The resulting graph is a spiral.

Notice that the resulting graph cannot be the result of a function of the form $y = f(x)$, as it does not pass the vertical line test, even though it resulted from a function giving r in terms of θ .



Although it is nice to see polar equations on polar grids, it is more common for polar graphs to be graphed on the Cartesian coordinate system, and so, the remainder of the polar equations will be graphed accordingly.

The spiral graph above on a Cartesian grid is shown here.

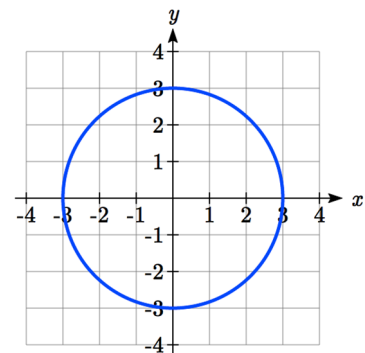


Example 6

Sketch a graph of the polar equation $r = 3$.

Recall that when a variable does not show up in the equation, it is saying that it does not matter what value that variable has; the output for the equation will remain the same. For example, the Cartesian equation $y = 3$ describes all the points where $y = 3$, no matter what the x values are, producing a horizontal line.

Likewise, this polar equation is describing all the points at a distance of 3 from the origin, no matter what the angle is, producing the graph of a **circle**.



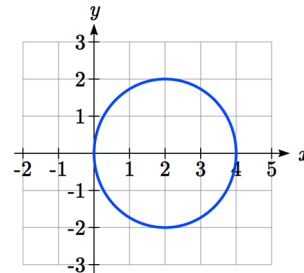
The normal settings on graphing calculators and software graph on the Cartesian coordinate system with y being a function of x , where the graphing utility asks for $f(x)$, or simply $y =$.

To graph polar equations, you may need to change the mode of your calculator to Polar. You will know you have been successful in changing the mode if you now have r as a function of θ , where the graphing utility asks for $r(\theta)$, or simply $r =$.

Example 7

Sketch a graph of the polar equation $r = 4\cos(\theta)$, and find an interval on which it completes one cycle.

While we could again create a table, plot the corresponding points, and connect the dots, we can also turn to technology to directly graph it. Using technology, we produce the graph shown here, a circle passing through the origin.



Since this graph appears to close a loop and repeat itself, we might ask what interval of θ values yields the entire graph. At $\theta = 0$, $r = 4\cos(0) = 4$, yielding the point $(4, 0)$. We want the next θ value when the graph returns to the point $(4, 0)$. Solving for when $x = 4$ is equivalent to solving $r\cos(\theta) = 4$.

$$r\cos(\theta) = 4$$

$$4\cos(\theta)\cos(\theta) = 4$$

$$\cos^2(\theta) = 1$$

$$\cos(\theta) = 1 \text{ or } \cos(\theta) = -1$$

$$\theta = 0 \text{ or } \theta = \pi$$

Substituting the equation for r gives

Dividing by 4 and simplifying

This has solutions when

Solving these gives solutions

This shows us at 0 radians we are at the point $(4, 0)$, and again at π radians we are at the point $(-4, 0)$ having finished one complete revolution.

The interval $0 \leq \theta < \pi$ yields one complete iteration of the circle.

Try it Now

3. Sketch a graph of the polar equation $r = 3\sin(\theta)$, and find an interval on which it completes one cycle.

The last few examples have all been circles. Next, we will consider two other “named” polar equations, **limaçons** and **roses**.

Example 8

Sketch a graph of the polar equation $r = 4\sin(\theta) + 2$. What interval of θ values corresponds to the inner loop?

This type of graph is called a **limaçon**.

Using technology, we can draw the graph. The inner loop begins and ends at the origin, where $r = 0$. We can solve for the θ values for which $r = 0$.

$$0 = 4\sin(\theta) + 2$$

$$-2 = 4\sin(\theta)$$

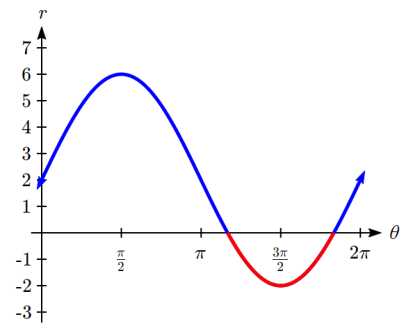
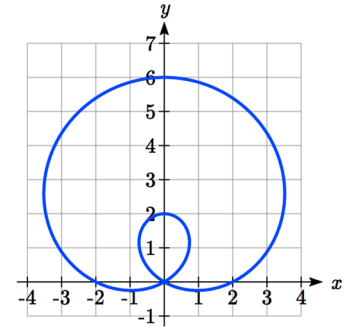
$$\sin(\theta) = -\frac{1}{2}$$

$$\theta = \frac{7\pi}{6} \text{ or } \theta = \frac{11\pi}{6}$$

This tells us that $r = 0$, so the graph passes through the origin, twice on the interval $[0, 2\pi)$.

The inner loop arises from the interval $\frac{7\pi}{6} \leq \theta \leq \frac{11\pi}{6}$.

This corresponds to where the function $r = 4\sin(\theta) + 2$ takes on negative values, as we could see if we graphed the function in the $r\theta$ plane.



Example 9

Sketch a graph of the polar equation $r = \cos(3\theta)$. What interval of θ values describes one small loop of the graph?

This type of graph is called a 3 leaf **rose**.

We can use technology to produce a graph. The interval $[0, \pi)$ yields one cycle of this function. As with the last problem, we can note that there is an interval on which one loop of this graph begins and ends at the origin, where $r = 0$. Solving for θ ,

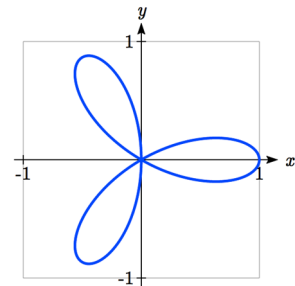
$$0 = \cos(3\theta)$$

$$\text{Substitute } u = 3\theta$$

$$0 = \cos(u)$$

$$u = \frac{\pi}{2} \text{ or } u = \frac{3\pi}{2} \text{ or } u = \frac{5\pi}{2}$$

Undo the substitution,



$$3\theta = \frac{\pi}{2} \quad \text{or} \quad 3\theta = \frac{3\pi}{2} \quad \text{or} \quad 3\theta = \frac{5\pi}{2}$$

$$\theta = \frac{\pi}{6} \quad \text{or} \quad \theta = \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{5\pi}{6}$$

There are 3 solutions on $0 \leq \theta < \pi$ which correspond to the 3 times the graph returns to the origin, but the first two solutions we solved for above are enough to conclude that one loop corresponds to the interval $\frac{\pi}{6} \leq \theta < \frac{\pi}{2}$.

If we wanted to get an idea of how the computer drew this graph, consider when $\theta = 0$. $r = \cos(3\theta) = \cos(0) = 1$, so the graph starts at $(1,0)$. As we found above, at $\theta = \frac{\pi}{6}$ and

$\theta = \frac{\pi}{2}$, the graph is at the origin. Looking at the equation,

notice that any angle in between $\frac{\pi}{6}$ and $\frac{\pi}{2}$, for example at

$\theta = \frac{\pi}{3}$, produces a negative r : $r = \cos\left(3 \cdot \frac{\pi}{3}\right) = \cos(\pi) = -1$.

θ	r	x	y
0	1	1	0
$\frac{\pi}{6}$	0	0	0
$\frac{\pi}{3}$	-1	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0	0	0

Notice that with a negative r value and an angle with terminal side in the first quadrant, the corresponding Cartesian point would be in the third quadrant. Since $r = \cos(3\theta)$ is negative

on $\frac{\pi}{6} \leq \theta < \frac{\pi}{2}$, this interval corresponds to the loop of the graph in the third quadrant.

Try it Now

4. Sketch a graph of the polar equation $r = \sin(2\theta)$. Would you call this function a **limaçon** or a **rose**?

Converting Equations

While many polar equations cannot be expressed nicely in Cartesian form (and vice versa), it can be beneficial to convert between the two forms, when possible. To do this we use the same relationships we used to convert points between coordinate systems.

Example 10

Rewrite the Cartesian equation $x^2 + y^2 = 6y$ as a polar equation.

We wish to eliminate x and y from the equation and introduce r and θ . Ideally, we would like to write the equation with r isolated, if possible, which represents r as a function of θ .

$$x^2 + y^2 = 6y$$

Remembering $x^2 + y^2 = r^2$ we substitute

$$r^2 = 6y$$

$y = r \sin(\theta)$ and so we substitute again

$$r^2 = 6r \sin(\theta)$$

Subtract $6r \sin(\theta)$ from both sides

$$r^2 - 6r \sin(\theta) = 0$$

Factor

$$r(r - 6 \sin(\theta)) = 0$$

Use the zero factor theorem

$$r = 6 \sin(\theta) \text{ or } r = 0$$

Since $r = 0$ is only a point, we reject that solution.

The solution $r = 6 \sin(\theta)$ is fairly similar to the one we graphed in Example 7. In fact, this equation describes a circle with bottom at the origin and top at the point $(0, 6)$.

Example 11

Rewrite the Cartesian equation $y = 3x + 2$ as a polar equation.

$$y = 3x + 2$$

Use $y = r \sin(\theta)$ and $x = r \cos(\theta)$

$$r \sin(\theta) = 3r \cos(\theta) + 2$$

Move all terms with r to one side

$$r \sin(\theta) - 3r \cos(\theta) = 2$$

Factor out r

$$r(\sin(\theta) - 3 \cos(\theta)) = 2$$

Divide

$$r = \frac{2}{\sin(\theta) - 3 \cos(\theta)}$$

In this case, the polar equation is more unwieldy than the Cartesian equation, but there are still times when this equation might be useful.

Example 12

Rewrite the polar equation $r = \frac{3}{1 - 2 \cos(\theta)}$ as a Cartesian equation.

We want to eliminate θ and r and introduce x and y . It is usually easiest to start by clearing the fraction and looking to substitute values that will eliminate θ .

$$r = \frac{3}{1 - 2\cos(\theta)}$$

Clear the fraction

$$r(1 - 2\cos(\theta)) = 3$$

Use $\cos(\theta) = \frac{x}{r}$ to eliminate θ

$$r\left(1 - 2\frac{x}{r}\right) = 3$$

Distribute and simplify

$$r - 2x = 3$$

Isolate the r

$$r = 3 + 2x$$

Square both sides

$$r^2 = (3 + 2x)^2$$

Use $x^2 + y^2 = r^2$

$$x^2 + y^2 = (3 + 2x)^2$$

When our entire equation has been changed from r and θ to x and y we can stop unless asked to solve for y or simplify.

In this example, if desired, the right side of the equation could be expanded and the equation simplified further. However, the equation cannot be written as a function in Cartesian form.

Try it Now

5. a. Rewrite the Cartesian equation in polar form: $y = \pm\sqrt{3 - x^2}$
- b. Rewrite the polar equation in Cartesian form: $r = 2\sin(\theta)$

Example 13

Rewrite the polar equation $r = \sin(2\theta)$ in Cartesian form.

$$r = \sin(2\theta)$$

Use the double angle identity for sine

$$r = 2\sin(\theta)\cos(\theta)$$

Use $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$

$$r = 2 \cdot \frac{x}{r} \cdot \frac{y}{r}$$

Simplify

$$r = \frac{2xy}{r^2}$$

Multiply by r^2

$$r^3 = 2xy$$

Since $x^2 + y^2 = r^2$, $r = \sqrt{x^2 + y^2}$

$$\left(\sqrt{x^2 + y^2}\right)^3 = 2xy$$

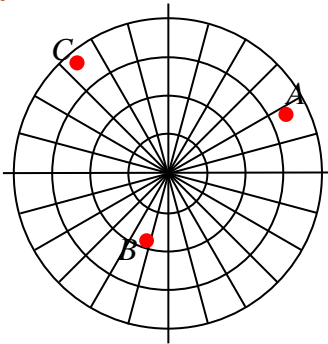
This equation could also be written as

$$(x^2 + y^2)^{3/2} = 2xy \quad \text{or} \quad x^2 + y^2 = (2xy)^{2/3}$$

Important Topics of This Section

Cartesian coordinate system
 Polar coordinate system
 Plotting points in polar coordinates
 Converting coordinates between systems
 Polar equations: Spirals, circles, limaçons and roses
 Converting equations between systems

Try it Now Answers



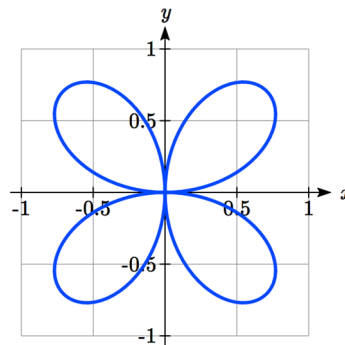
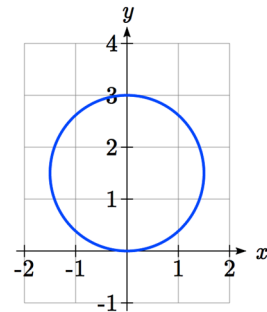
1.

2. a. $(r, \theta) = (2, \pi)$ converts to $(x, y) = (2 \cos(\pi), 2 \sin(\pi)) = (-2, 0)$

b. $(x, y) = (0, -4)$ converts to $(r, \theta) = \left(4, \frac{3\pi}{2}\right)$ or $\left(-4, \frac{\pi}{2}\right)$

3. $3 \sin(\theta) = 0$ at $\theta = 0$ and $\theta = \pi$.

It completes one cycle on the interval $0 \leq \theta < \pi$.



4. This is a 4-leaf rose.

5. a. $y = \pm\sqrt{3-x^2}$ can be rewritten as $x^2 + y^2 = 3$, and becomes $r = \sqrt{3}$

b. $r = 2 \sin(\theta)$. $r = 2 \frac{y}{r}$. $r^2 = 2y$. $x^2 + y^2 = 2y$

Section 8.2 Exercises

Convert the given polar coordinates to Cartesian coordinates.

1. $\left(7, \frac{7\pi}{6}\right)$ 2. $\left(6, \frac{3\pi}{4}\right)$ 3. $\left(4, \frac{7\pi}{4}\right)$ 4. $\left(9, \frac{4\pi}{3}\right)$

5. $\left(6, -\frac{\pi}{4}\right)$ 6. $\left(12, -\frac{\pi}{3}\right)$ 7. $\left(3, \frac{\pi}{2}\right)$ 8. $(5, \pi)$

9. $\left(-3, \frac{\pi}{6}\right)$ 10. $\left(-2, \frac{2\pi}{3}\right)$ 11. $(3, 2)$ 12. $(7, 1)$

Convert the given Cartesian coordinates to polar coordinates.

13. $(4, 2)$ 14. $(8, 8)$ 15. $(-4, 6)$ 16. $(-5, 1)$

17. $(3, -5)$ 18. $(6, -5)$ 19. $(-10, -13)$ 20. $(-4, -7)$

Convert the given Cartesian equation to a polar equation.

21. $x = 3$ 22. $y = 4$ 23. $y = 4x^2$ 24. $y = 2x^4$

25. $x^2 + y^2 = 4y$ 26. $x^2 + y^2 = 3x$ 27. $x^2 - y^2 = x$ 28. $x^2 - y^2 = 3y$

Convert the given polar equation to a Cartesian equation.

29. $r = 3 \sin(\theta)$ 30. $r = 4 \cos(\theta)$

31. $r = \frac{4}{\sin(\theta) + 7 \cos(\theta)}$ 32. $r = \frac{6}{\cos(\theta) + 3 \sin(\theta)}$

33. $r = 2 \sec(\theta)$ 34. $r = 3 \csc(\theta)$

35. $r = \sqrt{r \cos(\theta) + 2}$ 36. $r^2 = 4 \sec(\theta) \csc(\theta)$

Match each equation with one of the graphs shown.

37. $r = 2 + 2 \cos(\theta)$

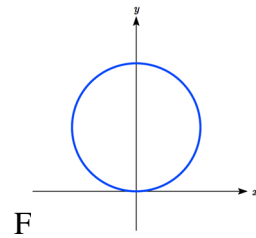
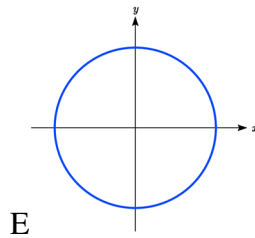
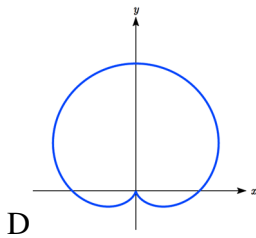
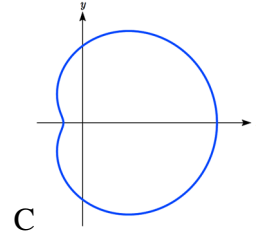
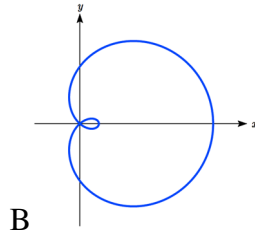
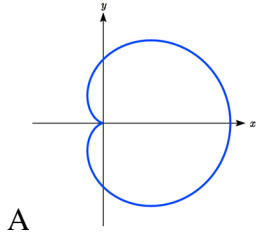
38. $r = 2 + 2 \sin(\theta)$

39. $r = 4 + 3 \cos(\theta)$

40. $r = 3 + 4 \cos(\theta)$

41. $r = 5$

42. $r = 2 \sin(\theta)$



Match each equation with one of the graphs shown.

43. $r = \log(\theta)$

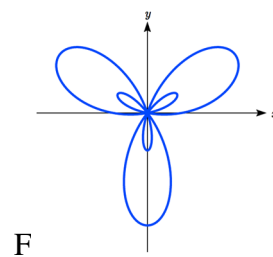
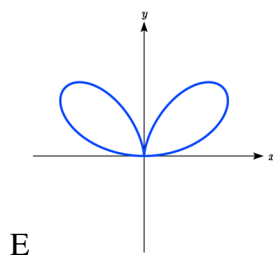
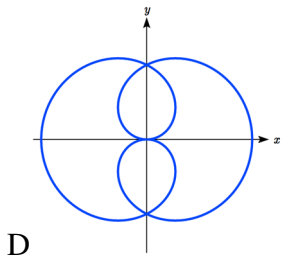
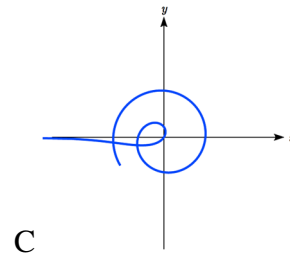
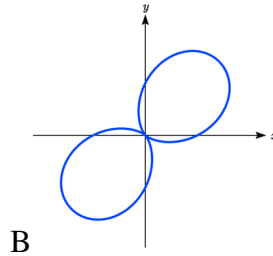
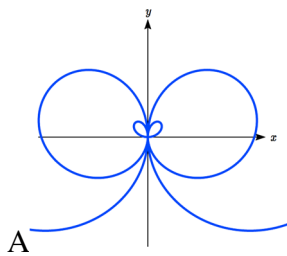
44. $r = \theta \cos(\theta)$

45. $r = \cos\left(\frac{\theta}{2}\right)$

46. $r = \sin(\theta) \cos^2(\theta)$

47. $r = 1 + 2 \sin(3\theta)$

48. $r = 1 + \sin(2\theta)$



Sketch a graph of the polar equation.

49. $r = 3 \cos(\theta)$

50. $r = 4 \sin(\theta)$

51. $r = 3 \sin(2\theta)$

52. $r = 4 \sin(4\theta)$

53. $r = 5 \sin(3\theta)$

54. $r = 4 \sin(5\theta)$

55. $r = 3 \cos(2\theta)$

56. $r = 4 \cos(4\theta)$

57. $r = 2 + 2 \cos(\theta)$

58. $r = 3 + 3 \sin(\theta)$

59. $r = 1 + 3 \sin(\theta)$

60. $r = 2 + 4 \cos(\theta)$

61. $r = 2\theta$

62. $r = \frac{1}{\theta}$

63. $r = 3 + \sec(\theta)$, a conchoid

64. $r = \frac{1}{\sqrt{\theta}}$, a lituus¹

65. $r = 2 \sin(\theta) \tan(\theta)$, a cissoid

66. $r = 2\sqrt{1 - \sin^2(\theta)}$, a hippopede

¹ This curve was the inspiration for the artwork featured on the cover of this book.