Chapter 6: Periodic Functions

In the previous chapter, the trigonometric functions were introduced as ratios of sides of a right triangle, and related to points on a circle. We noticed how the *x* and *y* values of the points did not change with repeated revolutions around the circle by finding coterminal angles. In this chapter, we will take a closer look at the important characteristics and applications of these types of functions, and begin solving equations involving them.

Section 6.1 Sinusoidal Graphs

The London Eye[1] is a huge Ferris wheel 135 meters (394 feet) tall in London, England, which completes one rotation every 30 minutes. When we look at the behavior of this Ferris wheel it is clear that it completes 1 cycle, or 1 revolution, and then repeats this revolution over and over again.

This is an example of a periodic function, because the Ferris wheel repeats its revolution or one cycle every 30 minutes, and so we say it has a period of 30 minutes.

In this section, we will work to sketch a graph of a rider's height above the ground over time and express this height as a function of time.

Periodic Functions
A **periodic function** is a function for which a specific horizontal shift, *P*, results in the original function: $f(x+P)=f(x)$ for all values of *x*. When this occurs we call the smallest such horizontal shift with $P > 0$ the **period** of the function.

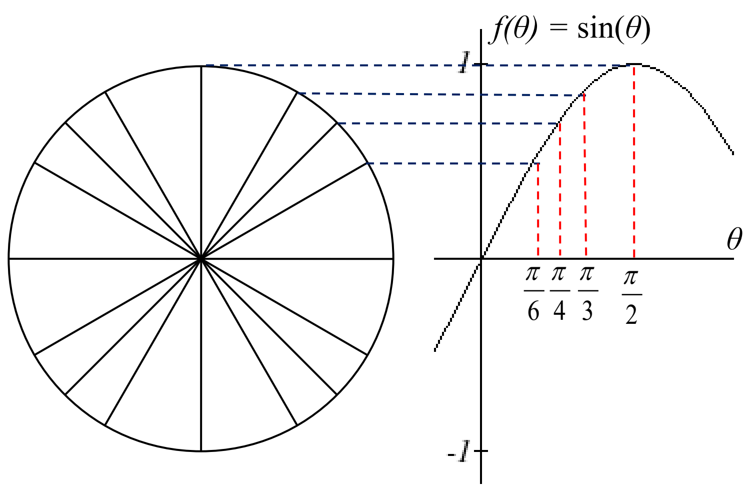
[1] London Eye photo by authors, 2010, CC-BY

You might immediately guess that there is a connection here to finding points on a circle, since the height above ground would correspond to the *y* value of a point on the circle. We can determine the *y* value by using the sine function. To get a better sense of this function's behavior, we can create a table of values we know, and use them to sketch a graph of the sine and cosine functions.

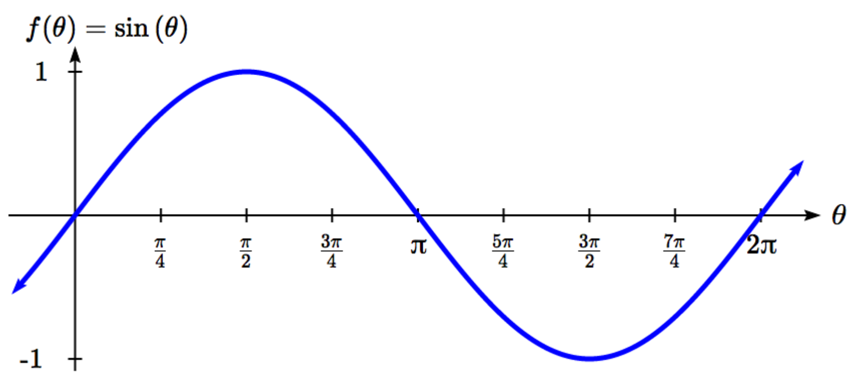
Listing some of the values for sine and cosine on a unit circle,

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
sin	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0

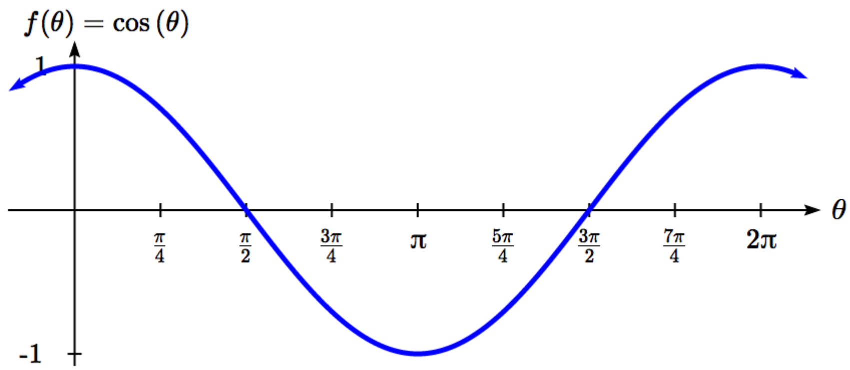
Here you can see how for each angle, we use the *y* value of the point on the circle to determine the output value of the sine function.



Plotting more points gives the full shape of the sine and cosine functions.



Notice how the sine values are positive between 0 and π, which correspond to the values of sine in quadrants 1 and 2 on the unit circle, and the sine values are negative between π and 2π, corresponding to quadrants 3 and 4.



Like the sine function we can track the value of the cosine function through the 4 quadrants of the unit circle as we place it on a graph.

Both of these functions are defined for all real numbers, since we can evaluate the sine and cosine of any angle. By thinking of sine and cosine as coordinates of points on a unit circle, it becomes clear that the range of both functions must be the interval $[-1, 1]$.

Domain and Range of Sine and Cosine
The domain of sine and cosine is all real numbers, $(-\infty, \infty)$. The range of sine and cosine is the interval [-1, 1].

Both these graphs are called **sinusoidal** graphs.

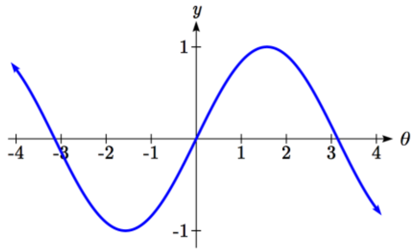
In both graphs, the shape of the graph begins repeating after 2π. Indeed, since any coterminal angles will have the same sine and cosine values, we could conclude that $\sin(\theta + 2\pi) = \sin(\theta)$ and $\cos(\theta + 2\pi) = \cos(\theta)$.

In other words, if you were to shift either graph horizontally by 2π, the resulting shape would be identical to the original function. Sinusoidal functions are a specific type of periodic function.

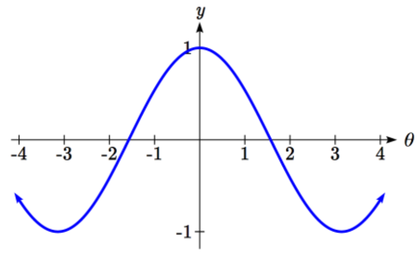
Period of Sine and Cosine
The periods of the sine and cosine functions are both 2π.

Looking at these functions on a domain centered at the vertical axis helps reveal symmetries.

sine



cosine



The sine function is symmetric about the origin, the same symmetry the cubic function has, making it an odd function. The cosine function is clearly symmetric about the y axis, the same symmetry as the quadratic function, making it an even function.

Negative Angle Identities
The sine is an odd function, symmetric about the *origin*, so $\sin(-\theta) = -\sin(\theta)$.
The cosine is an even function, symmetric about the y-axis, so $\cos(-\theta) = \cos(\theta)$.

These identities can be used, among other purposes, for helping with simplification and proving identities.

You may recall the cofunction identity from last chapter, $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$.

Graphically, this tells us that the sine and cosine graphs are horizontal transformations of each other. We can prove this by using the cofunction identity and the negative angle identity for cosine.

$$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right) = \cos\left(-\theta + \frac{\pi}{2}\right) = \cos\left(-\left(\theta - \frac{\pi}{2}\right)\right) = \cos\left(\theta - \frac{\pi}{2}\right)$$

Now we can clearly see that if we horizontally shift the cosine function to the right by $\pi/2$ we get the sine function.

Remember this shift is not representing the period of the function. It only shows that the cosine and sine function are transformations of each other.

Example 1

Simplify $\dfrac{\sin(-\theta)}{\tan(\theta)}$.

We start by using the negative angle identity for sine.

$\dfrac{-\sin(\theta)}{\tan(\theta)}$ Rewriting the tangent

$\dfrac{-\sin(\theta)}{\sin(\theta)/\cos(\theta)}$ Inverting and multiplying

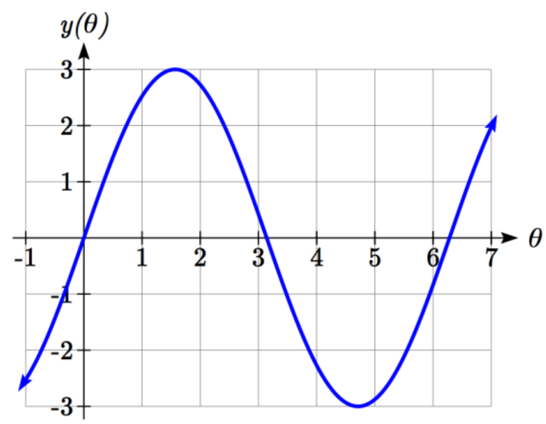
$-\sin(\theta)\cdot\dfrac{\cos(\theta)}{\sin(\theta)}$ Simplifying we get

$-\cos(\theta)$

Transforming Sine and Cosine

Example 2

A point rotates around a circle of radius 3. Sketch a graph of the y coordinate of the point.



Recall that for a point on a circle of radius r, the y coordinate of the point is $y = r\sin(\theta)$, so in this case, we get the equation $y(\theta) = 3\sin(\theta)$.

The constant 3 causes a vertical stretch of the y values of the function by a factor of 3.

Notice that the period of the function does not change.

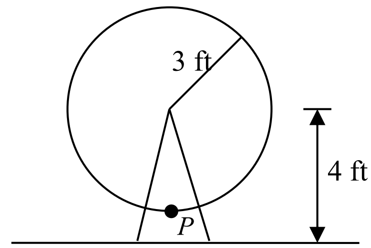
Since the outputs of the graph will now oscillate between -3 and 3, we say that the **amplitude** of the sine wave is 3.

Try it Now

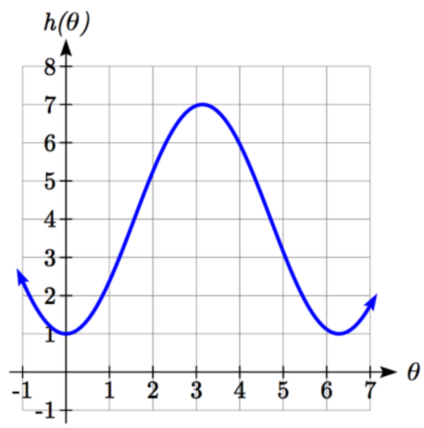
1. What is the amplitude of the function $f(\theta) = 7\cos(\theta)$? Sketch a graph of this function.

Example 3

A circle with radius 3 feet is mounted with its center 4 feet off the ground. The point closest to the ground is labeled *P*. Sketch a graph of the height above ground of the point *P* as the circle is rotated, then find a function that gives the height in terms of the angle of rotation.



Sketching the height, we note that it will start 1 foot above the ground, then increase up to 7 feet above the ground, and continue to oscillate 3 feet above and below the center value of 4 feet.



Although we could use a transformation of either the sine or cosine function, we start by looking for characteristics that would make one function easier to use than the other.

We decide to use a cosine function because it starts at the highest or lowest value, while a sine function starts at the middle value. A standard cosine starts at the highest value, and this graph starts at the lowest value, so we need to incorporate a vertical reflection.

Second, we see that the graph oscillates 3 above and below the center, while a basic cosine has an amplitude of one, so this graph has been vertically stretched by 3, as in the last example.

Finally, to move the center of the circle up to a height of 4, the graph has been vertically shifted up by 4. Putting these transformations together,

$$h(\theta) = -3\cos(\theta) + 4$$

Midline

The center value of a sinusoidal function, the value that the function oscillates above and below, is called the **midline** of the function, corresponding to a vertical shift.

The function $f(\theta) = \cos(\theta) + k$ has midline at $y = k$.

Try it Now

2. What is the midline of the function $f(\theta) = 3\cos(\theta) - 4$? Sketch a graph of the function.

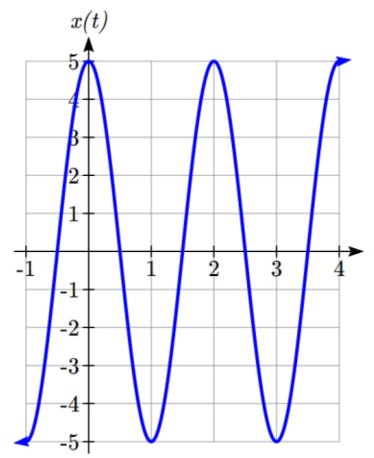
To answer the Ferris wheel problem at the beginning of the section, we need to be able to express our sine and cosine functions at inputs of time. To do so, we will utilize composition. Since the sine function takes an input of an angle, we will look for a function that takes time as an input and outputs an angle. If we can find a suitable $\theta(t)$ function, then we can compose this with our $f(\theta) = \cos(\theta)$ function to obtain a sinusoidal function of time: $f(t) = \cos(\theta(t))$.

Example 4

A point completes 1 revolution every 2 minutes around a circle of radius 5. Find the x coordinate of the point as a function of time, if it starts at (5, 0).

Normally, we would express the x coordinate of a point on a circle centered at the origin using $x = r\cos(\theta)$. Here we write the function $x(\theta) = 5\cos(\theta)$.

The rotation rate of 1 revolution every 2 minutes is an angular velocity. We can use this rate to find a formula for the angle as a function of time. The point begins at an angle of 0. Since the point rotates 1 revolution = 2π radians every 2 minutes, it rotates π radians every minute. After t minutes, it will have rotated:
$\theta(t) = \pi t$ radians



Composing this with the cosine function, we obtain a function of time.
$x(t) = 5\cos(\theta(t)) = 5\cos(\pi t)$

Notice that this composition has the effect of a horizontal compression, changing the period of the function.

To see how the period relates to the stretch or compression coefficient B in the equation $f(t) = \sin(Bt)$, note that the period will be the time it takes to complete one full revolution of a circle. If a point takes P minutes to complete 1 revolution, then the angular velocity is $\dfrac{2\pi \text{ radians}}{P \text{ minutes}}$. Then $\theta(t) = \dfrac{2\pi}{P}t$. Composing with a sine function,

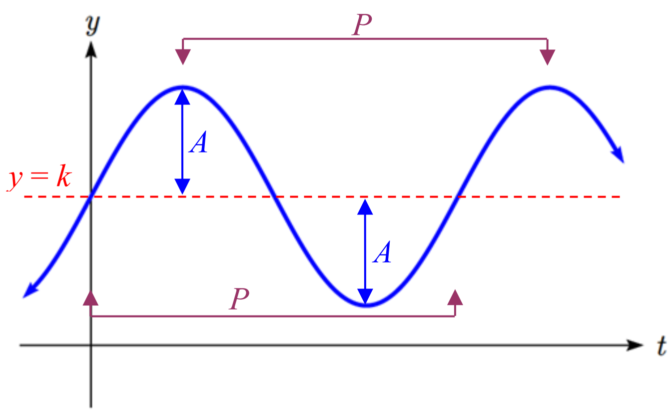
$$f(t) = \sin(\theta(t)) = \sin\left(\frac{2\pi}{P}t\right)$$

From this, we can determine the relationship between the coefficient B and the period: $B = \dfrac{2\pi}{P}$.

Notice that the stretch or compression coefficient B is a ratio of the "normal period of a sinusoidal function" to the "new period." If we know the stretch or compression coefficient B, we can solve for the "new period": $P = \frac{2\pi}{B}$.

Summarizing our transformations so far:

Transformations of Sine and Cosine
Given an equation in the form $f(t) = A\sin(Bt) + k$ or $f(t) = A\cos(Bt) + k$ A is the vertical stretch, and is the **amplitude** of the function. B is the horizontal stretch/compression, and is related to the **period, *P***, by $P = \frac{2\pi}{B}$. k is the vertical shift and determines the **midline** of the function.



Example 5

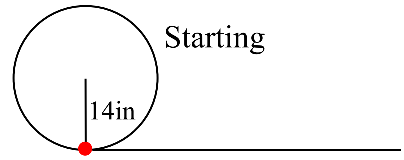
What is the period of the function $f(t) = \sin\left(\frac{\pi}{6}t\right)$?

Using the relationship above, the stretch/compression factor is $B = \frac{\pi}{6}$, so the period will be $P = \frac{2\pi}{B} = \frac{2\pi}{\frac{\pi}{6}} = 2\pi \cdot \frac{6}{\pi} = 12$.

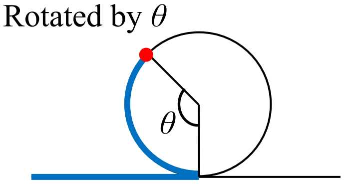
While it is common to compose sine or cosine with functions involving time, the composition can be done so that the input represents any reasonable quantity.

Example 6

A bicycle wheel with radius 14 inches has the bottom-most point on the wheel marked in red. The wheel then begins rolling down the street. Write a formula for the height above ground of the red point after the bicycle has travelled x inches.



The height of the point begins at the lowest value, 0, increases to the highest value of 28 inches, and continues to oscillate above and below a center height of 14 inches. In terms of the angle of rotation, θ:
$h(\theta) = -14\cos(\theta) + 14$



In this case, x is representing a linear distance the wheel has travelled, corresponding to an arclength along the circle. Since arclength and angle can be related by $s = r\theta$, in this case we can write $x = 14\theta$, which allows us to express the angle in terms of x:

$$\theta(x) = \frac{x}{14}$$

Composing this with our cosine-based function from above,

$$h(x) = h(\theta(x)) = -14\cos\left(\frac{x}{14}\right) + 14 = -14\cos\left(\frac{1}{14}x\right) + 14$$

The period of this function would be $P = \dfrac{2\pi}{B} = \dfrac{2\pi}{\frac{1}{14}} = 2\pi \cdot 14 = 28\pi$, the circumference of the circle. This makes sense – the wheel completes one full revolution after the bicycle has travelled a distance equivalent to the circumference of the wheel.

Example 7

Determine the midline, amplitude, and period of the function $f(t) = 3\sin(2t) + 1$.

The amplitude is 3

The period is $P = \dfrac{2\pi}{B} = \dfrac{2\pi}{2} = \pi$

The midline is at $y = 1$

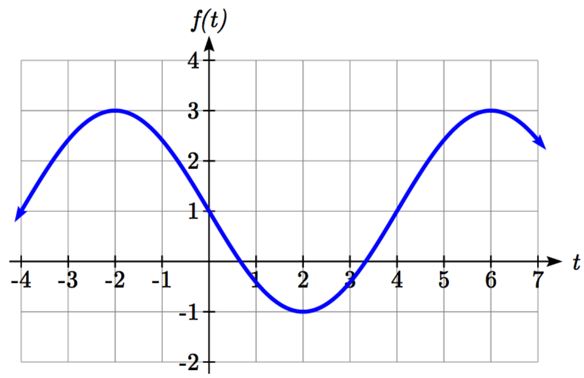
Amplitude, midline, and period, when combined with vertical flips, allow us to write equations for a variety of sinusoidal situations.

Try it Now

3. If a sinusoidal function starts on the midline at point (0,3), has an amplitude of 2, and a period of 4, write a formula for the function

Example 8

Find a formula for the sinusoidal function graphed here.



The graph oscillates from a low of -1 to a high of 3, putting the midline at $y = 1$, halfway between.

The amplitude will be 2, the distance from the midline to the highest value (or lowest value) of the graph.

The period of the graph is 8. We can measure this from the first peak at $x = -2$ to the second at $x = 6$. Since the period is 8, the stretch/compression factor we will use will be

$$B = \frac{2\pi}{P} = \frac{2\pi}{8} = \frac{\pi}{4}$$

At $x = 0$, the graph is at the midline value, which tells us the graph can most easily be represented as a sine function. Since the graph then decreases, this must be a vertical reflection of the sine function. Putting this all together,

$$f(t) = -2\sin\left(\frac{\pi}{4}t\right) + 1$$

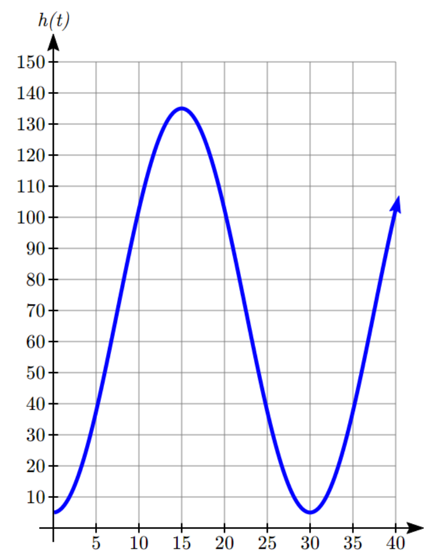
With these transformations, we are ready to answer the Ferris wheel problem from the beginning of the section.

Example 9

The London Eye is a huge Ferris wheel in London, England, which completes one rotation every 30 minutes. The diameter of the wheel is 120 meters, but the passenger capsules sit outside the wheel. Suppose the diameter at the capsules is 130 meters, and riders board from a platform 5 meters above the ground. Express a rider's height above ground as a function of time in minutes.

It can often help to sketch a graph of the situation before trying to find the equation.

With a diameter of 130 meters, the wheel has a radius of 65 meters. The height will oscillate with amplitude of 65 meters above and below the center.



Passengers board 5 meters above ground level, so the center of the wheel must be located 65 + 5 = 70 meters above ground level. The midline of the oscillation will be at 70 meters.

The wheel takes 30 minutes to complete 1 revolution, so the height will oscillate with period of 30 minutes.

Lastly, since the rider boards at the lowest point, the height will start at the smallest value and increase, following the shape of a flipped cosine curve.
Putting these together:
Amplitude: 65
Midline: 70
Period: 30, so $B = \dfrac{2\pi}{30} = \dfrac{\pi}{15}$
Shape: negative cosine

An equation for the rider's height would be
$h(t) = -65\cos\left(\dfrac{\pi}{15}t\right) + 70$

Try it Now

4. The Ferris wheel at the Puyallup Fair[2] has a diameter of about 70 feet and takes 3 minutes to complete a full rotation. Passengers board from a platform 10 feet above the ground. Write an equation for a rider's height above ground over time.

While these transformations are sufficient to represent many situations, occasionally we encounter a sinusoidal function that does not have a vertical intercept at the lowest point, highest point, or midline. In these cases, we need to use horizontal shifts. Since we are combining horizontal shifts with horizontal stretches, we need to be careful. Recall that when the inside of the function is factored, it reveals the horizontal shift.

[2] Photo by photogirl7.1, http://www.flickr.com/photos/kitkaphotogirl/432886205/sizes/z/, CC-BY

Horizontal Shifts of Sine and Cosine

Given an equation in the form $f(t) = A\sin\left(B(t-h)\right)+k$ or $f(t) = A\cos\left(B(t-h)\right)+k$

h is the horizontal shift of the function

Example 10

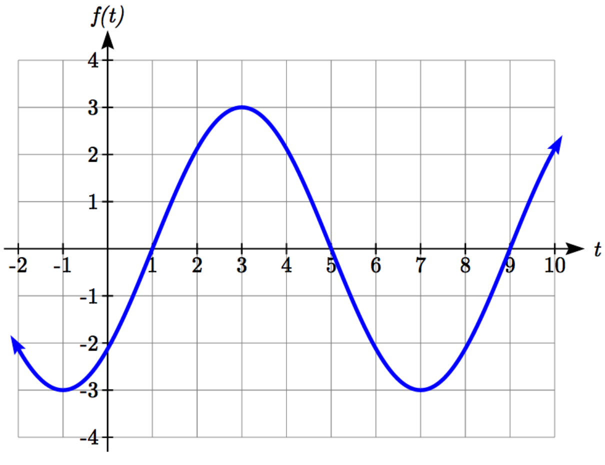
Sketch a graph of $f(t) = 3\sin\left(\frac{\pi}{4}t - \frac{\pi}{4}\right)$.

To reveal the horizontal shift, we first need to factor inside the function:

$$f(t) = 3\sin\left(\frac{\pi}{4}(t-1)\right)$$

This graph will have the shape of a sine function, starting at the midline and increasing, with an amplitude of 3. The period of the graph will be $P = \frac{2\pi}{B} = \frac{2\pi}{\frac{\pi}{4}} = 2\pi \cdot \frac{4}{\pi} = 8$.

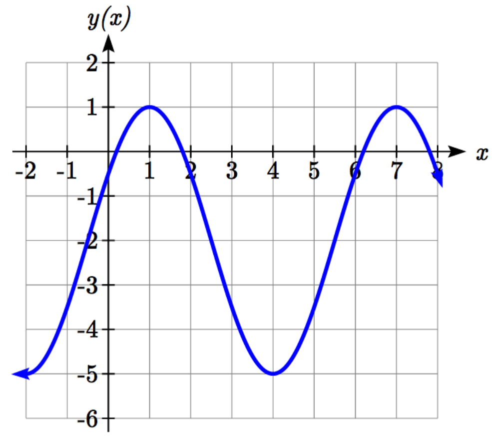
Finally, the graph will be shifted to the right by 1.



In some physics and mathematics books, you will hear the horizontal shift referred to as **phase shift**. In other physics and mathematics books, they would say the phase shift of the equation above is $\frac{\pi}{4}$, the value in the unfactored form. Because of this ambiguity, we will not use the term phase shift any further, and will only talk about the horizontal shift.

Example 11

Find a formula for the function graphed here.



With highest value at 1 and lowest value at -5, the midline will be halfway between at -2.

The distance from the midline to the highest or lowest value gives an amplitude of 3.

The period of the graph is 6, which can be measured from the peak at $x = 1$ to the next peak at $x = 7$, or from the distance between the lowest points. This gives $B = \dfrac{2\pi}{P} = \dfrac{2\pi}{6} = \dfrac{\pi}{3}$.

For the shape and shift, we have more than one option. We could either write this as:
- A cosine shifted 1 to the right
- A negative cosine shifted 2 to the left
- A sine shifted ½ to the left
- A negative sine shifted 2.5 to the right

While any of these would be fine, the cosine shifts are easier to work with than the sine shifts in this case, because they involve integer values. Writing these:

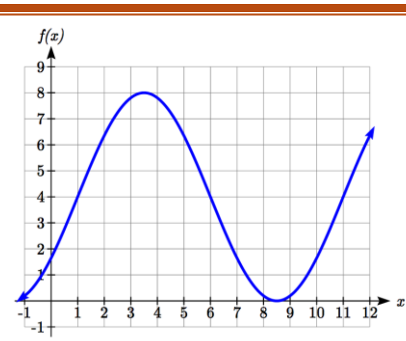
$$y(x) = 3\cos\left(\frac{\pi}{3}(x-1)\right) - 2 \qquad \text{or}$$

$$y(x) = -3\cos\left(\frac{\pi}{3}(x+2)\right) - 2$$

Again, these functions are equivalent, so both yield the same graph.

Try it Now

5. Write a formula for the function graphed here.



Important Topics of This Section

Periodic functions
Sine and cosine function from the unit circle
Domain and range of sine and cosine functions
Sinusoidal functions
Negative angle identity
Simplifying expressions
Transformations
  Amplitude
  Midline
  Period
  Horizontal shifts

Try it Now Answers

1. 7

2. -4

3. $f(x) = 2\sin\left(\frac{\pi}{2}x\right) + 3$

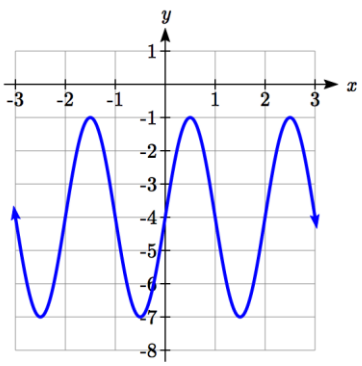
4. $h(t) = -35\cos\left(\frac{2\pi}{3}t\right) + 45$

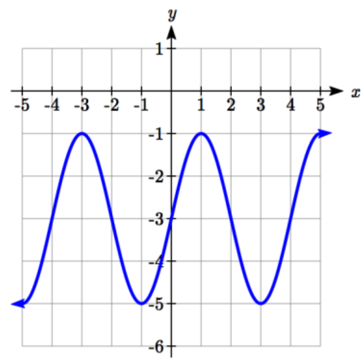
5. Two possibilities: $f(x) = 4\cos\left(\frac{\pi}{5}(x - 3.5)\right) + 4$ or $f(x) = 4\sin\left(\frac{\pi}{5}(x - 1)\right) + 4$

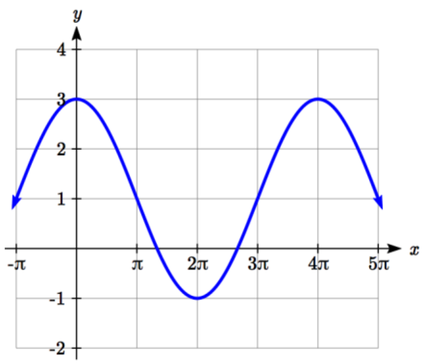
Section 6.1 Exercises

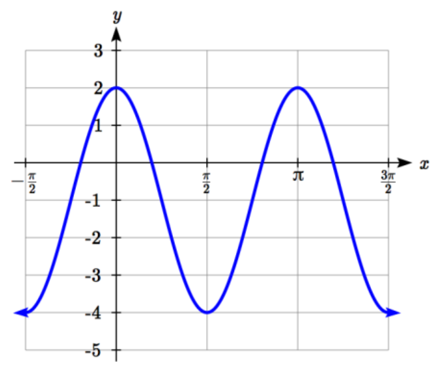
1. Sketch a graph of $f(x) = -3\sin(x)$.
2. Sketch a graph of $f(x) = 4\sin(x)$.
3. Sketch a graph of $f(x) = 2\cos(x)$.
4. Sketch a graph of $f(x) = -4\cos(x)$.

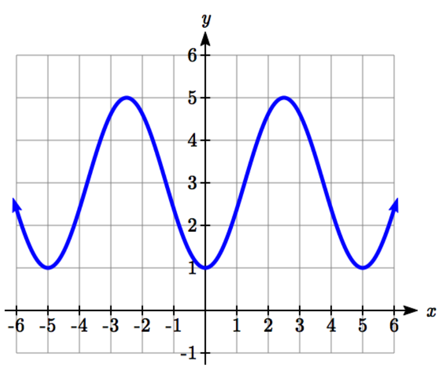
For the graphs below, determine the amplitude, midline, and period, then find a formula for the function.

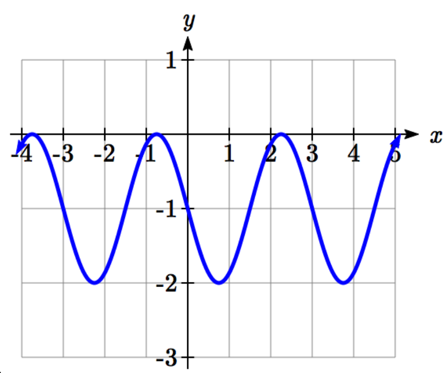
5. 

6. 

7. 

8. 

9. 

10. 

For each of the following equations, find the amplitude, period, horizontal shift, and midline.

11. $y = 3\sin(8(x+4))+5$

12. $y = 4\sin\left(\frac{\pi}{2}(x-3)\right)+7$

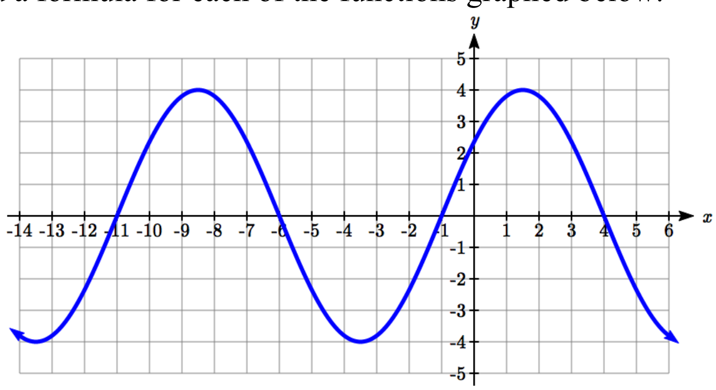
13. $y = 2\sin(3x-21)+4$

14. $y = 5\sin(5x+20)-2$

15. $y = \sin\left(\frac{\pi}{6}x+\pi\right)-3$

16. $y = 8\sin\left(\frac{7\pi}{6}x+\frac{7\pi}{2}\right)+6$

Find a formula for each of the functions graphed below.



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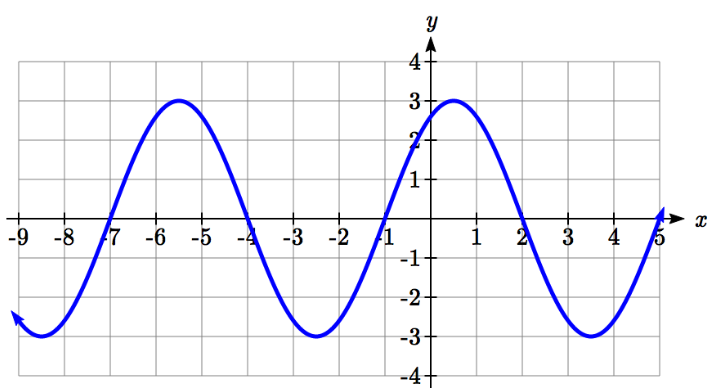
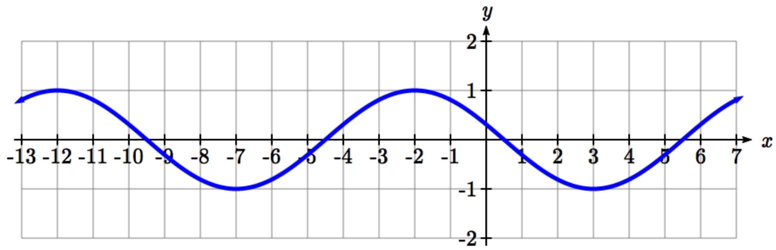
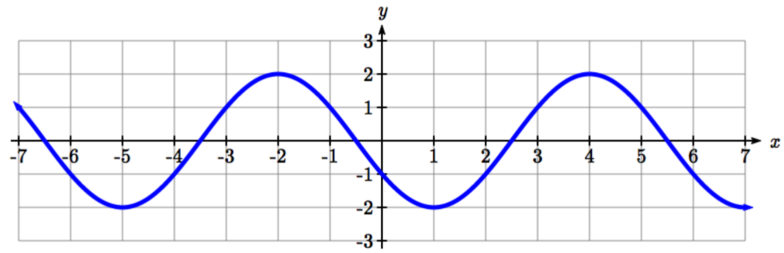


18.

19. 

20. 

21. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature is 50 degrees at midnight and the high and low temperature during the day are 57 and 43 degrees, respectively. Assuming t is the number of hours since midnight, find a function for the temperature, D, in terms of t.

22. Outside temperature over the course of a day can be modeled as a sinusoidal function. Suppose you know the temperature is 68 degrees at midnight and the high and low temperature during the day are 80 and 56 degrees, respectively. Assuming t is the number of hours since midnight, find a function for the temperature, D, in terms of t.

23. A Ferris wheel is 25 meters in diameter and boarded from a platform that is 1 meters above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 10 minutes. The function $h(t)$ gives your height in meters above the ground t minutes after the wheel begins to turn.
    a. Find the amplitude, midline, and period of $h(t)$.
    b. Find a formula for the height function $h(t)$.
    c. How high are you off the ground after 5 minutes?

24. A Ferris wheel is 35 meters in diameter and boarded from a platform that is 3 meters above the ground. The six o'clock position on the Ferris wheel is level with the loading platform. The wheel completes 1 full revolution in 8 minutes. The function $h(t)$ gives your height in meters above the ground t minutes after the wheel begins to turn.
    a. Find the amplitude, midline, and period of $h(t)$.
    b. Find a formula for the height function $h(t)$.
    c. How high are you off the ground after 4 minutes?