Chapter 8: Probability

Section 8.1 Concepts of Probability

Introduction

The probability of a specified event is the chance or likelihood that it will occur. There are several ways of viewing probability. One would be **experimental** in nature, where we repeatedly conduct an experiment. Suppose we flipped a coin over and over and over again and it came up heads about half of the time; we would expect that in the future whenever we flipped the coin it would turn up heads about half of the time. When a weather reporter says "there is a 10% chance of rain tomorrow," she is basing that on prior evidence; that out of all days with similar weather patterns, it has rained on 1 out of 10 of those days.

Another view would be **subjective** in nature, in other words an educated guess. If someone asked you the probability that the Seattle Mariners would win their next baseball game, it would be impossible to conduct an experiment where the same two teams played each other repeatedly, each time with the same starting lineup and starting pitchers, each starting at the same time of day on the same field under the precisely the same conditions. Since there are so many variables to take into account, someone familiar with baseball and with the two teams involved might make an educated guess that there is a 75% chance they will win the game; that is, *if* the same two teams were to play each other repeatedly under identical conditions, the Mariners would win about three out of every four games. But this is just a guess, with no way to verify its accuracy, and depending upon how educated the educated guesser is, a subjective probability may not be worth very much.

We will return to the experimental and subjective probabilities from time to time, but in this course we will mostly be concerned with **theoretical** probability, which is defined as follows: Suppose there is a situation with *n* <u>equally likely</u> possible outcomes and that *m* of those *n* outcomes correspond to a particular event; then the **probability** of that event is defined as $\frac{m}{n}$.

Basic Concepts

If you roll a die, pick a card from deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology.

Events and Outcomes

The result of an experiment is called an **outcome**.

An **event** is any particular outcome or group of outcomes.

A **simple event** is an event that cannot be broken down further

The **sample space** is the set of all possible simple events.

Example 1

If we roll a standard 6-sided die, describe the sample space and some simple events.

The sample space is the set of all possible simple events: {1,2,3,4,5,6}

Some examples of simple events:
We roll a 1
We roll a 5

Some compound events:
We roll a number bigger than 4
We roll an even number

Two dice   One die

Basic Probability

Given that all outcomes are equally likely, we can compute the probability of an event E using this formula:

$$P(E) = \frac{\text{Number of outcomes corresponding to the event E}}{\text{Total number of equally - likely outcomes}}$$

Example 2

If we roll a 6-sided die, calculate
a) P(rolling a 1)
b) P(rolling a number bigger than 4)

Recall that the sample space is {1,2,3,4,5,6}

a) There is one outcome corresponding to "rolling a 1", so the probability is $\frac{1}{6}$

b) There are two outcomes bigger than a 4, so the probability is $\frac{2}{6} = \frac{1}{3}$

Probabilities are essentially fractions, and can be reduced to lower terms like fractions.

Example 3

Let's say you have a bag with 20 cherries, 14 sweet and 6 sour. If you pick a cherry at random, what is the probability that it will be sweet?

There are 20 possible cherries that could be picked, so the number of possible outcomes is 20. Of these 20 possible outcomes, 14 are favorable (sweet), so the probability that the cherry will be sweet is $\dfrac{14}{20} = \dfrac{7}{10}$.

There is one potential complication to this example, however. It must be assumed that the probability of picking any of the cherries is the same as the probability of picking any other. This wouldn't be true if (let us imagine) the sweet cherries are smaller than the sour ones. (The sour cherries would come to hand more readily when you sampled from the bag.) Let us keep in mind, therefore, that when we assess probabilities in terms of the ratio of favorable to all potential cases, we rely heavily on the assumption of equal probability for all outcomes.

Try it Now

1. At some random moment, you look at your clock and note the minutes reading.
a. What is probability the minutes reading is 15?
b. What is the probability the minutes reading is 15 or less?

Cards

A standard deck of 52 playing cards consists of four **suits** (hearts, spades, diamonds and clubs). Spades and clubs are black while hearts and diamonds are red. Each suit contains 13 cards, each of a different **rank**: an Ace (which in many games functions as both a low card and a high card), cards numbered 2 through 10, a Jack, a Queen and a King.

Example 4

Compute the probability of randomly drawing one card from a deck and getting an Ace.

There are 52 cards in the deck and 4 Aces so $P(Ace) = \dfrac{4}{52} = \dfrac{1}{13} \approx 0.0769$

We can also think of probabilities as percents: There is a 7.69% chance that a randomly selected card will be an Ace.

Notice that the smallest possible probability is 0 – if there are no outcomes that correspond with the event. The largest possible probability is 1 – if all possible outcomes correspond with the event.

Certain and Impossible events

An impossible event has a probability of 0.
A certain event has a probability of 1.
The probability of any event must be $0 \le P(E) \le 1$

As you're working through this chapter, *if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work.*

Complementary Events

Now let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a six-sided die and first compute the probability of rolling a six: the answer is $P(\text{six}) = 1/6$. Now consider the probability that we do *not* roll a six: there are 5 outcomes that are not a six, so the answer is $P(\text{not a six}) = \dfrac{5}{6}$. Notice that

$$P(\text{six}) + P(\text{not a six}) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$$

This is not a coincidence. Consider a generic situation with n possible outcomes and an event E that corresponds to m of these outcomes. Then the remaining $n - m$ outcomes correspond to E not happening, thus

$$P(\text{not } E) = \frac{n-m}{n} = \frac{n}{n} - \frac{m}{n} = 1 - \frac{m}{n} = 1 - P(E)$$

Complement of an Event

The **complement** of an event is the event "E doesn't happen"
The notation $\overline{E}$ is used for the complement of event E.
We can compute the probability of the complement using $P(\overline{E}) = 1 - P(E)$
Notice also that $P(E) = 1 - P(\overline{E})$

Example 5

If you pull a random card from a deck of playing cards, what is the probability it is not a heart?

There are 13 hearts in the deck, so $P(\text{heart}) = \dfrac{13}{52} = \dfrac{1}{4}$.

The probability of *not* drawing a heart is the complement:

$$P(\text{not heart}) = 1 - P(\text{heart}) = 1 - \frac{1}{4} = \frac{3}{4}$$

Sometimes you will see probabilities expressed as **odds**.

Odds

Odds of an event are typically expressed in the form *A:B*

$\big($Number of outcomes corresponding to the event E$\big):\big($Number of outcomes corresponding to $\overline{\text{E}}\big)$

Example 6

If you pull a random card from a deck of playing cards, what are the odds is an Ace?

There are 4 Aces in the deck, and 48 cards that are not Aces, so the odds would be: 4:48, or 1:12

Notice how this is different than a probability – with probabilities we use the number of successes out of the total number of possible outcomes, while with odds we use the number of successes compared to the number of failures. As another example, if we flipped a coin, the odds of getting a heads would be 1:1.

Probability of two independent events

Example 7

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin and a 6 on the die.

We could list all possible outcomes: {H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6}. Notice there are $2 \cdot 6 = 12$ total outcomes. Out of these, only 1 is the desired outcome, so the probability is $\frac{1}{12}$.

The prior example was looking at two independent events.

Independent Events

Events A and B are **independent events** if the probability of Event B occurring is the same whether or not Event A occurs.

Example 8

Are these events independent?
a) A fair coin is tossed two times. The two events are (1) first toss is a head and (2) second toss is a head.

b) The two events (1) "It will rain tomorrow in Houston" and (2) "It will rain tomorrow in Galveston" (a city near Houston).

c) You draw a card from a deck, then draw a second card without replacing the first.

a) The probability that a head comes up on the second toss is 1/2 regardless of whether or not a head came up on the first toss, so these events are independent.

b) These events are not independent because it is more likely that it will rain in Galveston on days it rains in Houston than on days it does not.

c) The probability of the second card being red depends on whether the first card is red or not, so these events are not independent.

When two events are independent, the probability of both occurring is the product of the probabilities of the individual events.

$P(A$ and $B)$ for independent events

If events A and B are independent, then the probability of both A and B occurring is

$P(A$ and $B) = P(A) \cdot P(B)$

where $P(A$ and $B)$ is the probability of events A and B both occurring, $P(A)$ is the probability of event A occurring, and $P(B)$ is the probability of event B occurring

If you look back at the coin and die example from earlier, you can see how the number of outcomes of the first event multiplied by the number of outcomes in the second event multiplied to equal the total number of possible outcomes in the combined event.

Example 9

In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you randomly reach in and pull out a pair of socks and a tee shirt, what is the probability both are white?

The probability of choosing a white pair of socks is $\dfrac{6}{10}$.

The probability of choosing a white tee shirt is $\dfrac{3}{7}$.

The probability of both being white is $\dfrac{6}{10} \cdot \dfrac{3}{7} = \dfrac{18}{70} = \dfrac{9}{35}$

Example 10

The manufacturing process for a certain product has a 0.2% defect rate, meaning 2 products out of 1000 is defective on average. If two items are pulled randomly off the assembly line, what's the probability both are defective?

The probability of each being defective is independent, so the probability of both defective is $\dfrac{2}{1000} \cdot \dfrac{2}{1000} = \dfrac{4}{1,000,000} = \dfrac{1}{250,000}$

Try it Now

2. A card is pulled a deck of cards and noted. The card is then replaced, the deck is shuffled, and a second card is removed and noted. What is the probability that both cards are Aces?

The previous examples looked at the probability of *both* events occurring. Now we will look at the probability of *either* event occurring.

Example 11

Suppose we flipped a coin and rolled a die, and wanted to know the probability of getting a head on the coin *or* a 6 on the die.

Here, there are still 12 possible outcomes: {H1,H2,H3,H4,H5,H6,T1,T2,T3,T4,T5,T6}

By simply counting, we can see that 7 of the outcomes have a head on the coin *or* a 6 on the die *or* both – we use *or* inclusively here (these 7 outcomes are H1, H2, H3, H4, H5, H6, T6), so the probability is $\dfrac{7}{12}$. How could we have found this from the individual probabilities?

As we would expect, $\frac{1}{2}$ of these outcomes have a head, and $\frac{1}{6}$ of these outcomes have a 6 on the die. If we add these, $\frac{1}{2} + \frac{1}{6} = \frac{6}{12} + \frac{2}{12} = \frac{8}{12}$, which is not the correct probability. Looking at the outcomes we can see why: the outcome H6 would have been counted twice, since it contains both a head and a 6; the probability of both a head *and* rolling a 6 is $\frac{1}{12}$.

If we subtract out this double count, we have the correct probability: $\frac{8}{12} - \frac{1}{12} = \frac{7}{12}$.

P(A or B)

The probability of either *A* or *B* occurring (or both) is

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

Example 12

Suppose we draw one card from a standard deck. What is the probability that we get a Queen or a King?

There are 4 Queens and 4 Kings in the deck, hence 8 outcomes corresponding to a Queen or King out of 52 possible outcomes. Thus the probability of drawing a Queen or a King is:

$P(\text{King or Queen}) = \frac{8}{52}$

Note that in this case, there are no cards that are both a Queen and a King, so $P(\text{King and Queen}) = 0$. Using our probability rule, we could have said:

$P(\text{King or Queen}) = P(\text{King}) + P(\text{Queen}) - P(\text{King and Queen}) = \frac{4}{52} + \frac{4}{52} - 0 = \frac{8}{52}$

In the last example, the events were **mutually exclusive**, so $P(A \text{ or } B) = P(A) + P(B)$.

Example 13

Suppose we draw one card from a standard deck. What is the probability that we get a red card or a King?

Half the cards are red, so $P(\text{red}) = \frac{26}{52}$

There are four kings, so $P(\text{King}) = \frac{4}{52}$

There are two red kings, so $P(\text{Red and King}) = \dfrac{2}{52}$

We can then calculate

$$P(\text{Red or King}) = P(\text{Red}) + P(\text{King}) - P(\text{Red and King}) = \dfrac{26}{52} + \dfrac{4}{52} - \dfrac{2}{52} = \dfrac{28}{52}$$

3. In your drawer you have 10 pairs of socks, 6 of which are white, and 7 tee shirts, 3 of which are white. If you reach in and randomly grab a pair of socks and a tee shirt, what the probability at least one is white?

Example 14

The table below shows the number of survey subjects who have received and not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:
a) Has a red car *and* got a speeding ticket
b) Has a red car *or* got a speeding ticket.

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is $\dfrac{15}{665} \approx 0.0226$.

Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of probabilities of each one occurring.

We could answer this question by simply adding up the numbers: 15 people with red cars and speeding tickets + 135 with red cars but no ticket + 45 with a ticket but no red car = 195 people. So the probability is $\dfrac{195}{665} \approx 0.2932$.

We also could have found this probability by:
P(had a red car) + P(got a speeding ticket) – P(had a red car and got a speeding ticket)
$$= \dfrac{150}{665} + \dfrac{60}{665} - \dfrac{15}{665} = \dfrac{195}{665} .$$

Important Topics of this Section

Experimental, subjective, and theoretic probability
Events, sample space
Basic probability
Complementary events
Odds
Probability for independent events
Computing an "or" probability

Try it Now Answers

1. There are 60 possible readings, from 00 to 59. a. $\dfrac{1}{60}$ b. $\dfrac{16}{60}$ (counting 00 through 15)

2. Since the second draw is made after replacing the first card, these events are independent. The probability of an ace on each draw is $\dfrac{4}{52} = \dfrac{1}{13}$, so the probability of an Ace on both draws is $\dfrac{1}{13} \cdot \dfrac{1}{13} = \dfrac{1}{169}$

3. P(white sock or white tee) $= \dfrac{6}{10} + \dfrac{3}{7} - \dfrac{9}{35} = \dfrac{27}{35}$